Math 325K Fall 2018
Practice problem set
Solutions

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This problem set is to help you prepare for the final exam. Most final exam problems would be of similar style as some problems in this set.

1. True/False: each of the following arguments is either true or false.

(1) If the premise of a conditional statement is false, then the statement itself is false too.

Solution. $\boxed{False}$. *If the premise is false, no matter what the conclusion is, the statement itself is always true.*

(2) Let A, B be two sets. The Cartesian products $A \times B$ and $B \times A$ are NOT always the same.

Solution. $\boxed{True}$. *Since the elements in Cartesian products are* ordered *pairs,* $A \times B$ *and* $B \times A$ *are different in general.*

(3) Let p, q be two statements. If $p \vee q$ is false, then $p \wedge q$ is false too.

Solution. $\boxed{True}$. *If* $p \vee q$ *is false, then both* p, q *must be false and thus* $p \wedge q$ *is false.*

(4) The inverse and the converse of the same conditional statement are logically equivalent.

Solution. $\boxed{True}$. *Because they are contrapositive.*

(5) To disprove a universal statement, one counterexample is enough.

Solution. $\boxed{True}$. *One counterexample literally means that the statement is not universally true.*

(6) In multi-quantified statements, the statement remains the same if we change the order of the quantifiers.

Solution. $\boxed{False}$. *The order of quantifiers matters.*

(7) Every positive integer is either prime or composite.

Solution. $\boxed{False}$. 1 *is the only counterexample.*

(8) If the product of two integers a, b is even, then at least one of them is also even.

Solution. $\boxed{True}$. *The contraposition is obvious that the product of two odd numbers is still odd.*

(9) The difference of any two rational numbers is still rational.

Solution. $\boxed{True}$. $\frac{a}{b} - \frac{c}{d} = \frac{ad-bc}{bd}$.

(10) Suppose one justifies the statement $P(a)$ in the basis step of a proof by induction, then $a \geq 0$.

Solution. $\boxed{False}$. *a could be any integer including negative ones.*

(11) $\sum_{i=1}^{100} i^2 = \sum_{j=1}^{100} j^2$.

Solution. $\boxed{True}$. *The index i is dummy, which means it could be replaced by any other symbol as long as all occurrences of it are replaced by the same new symbol.*

(12) Let $\{F_n\}$ be the Fibonacci sequence. Then $F_{n+1} \leq 2F_n$ for all positive integers n.

Solution. $\boxed{True}$. *We use induction to prove the following statements one by one: $F_n > 0$ for all $n \geq 0$; $F_{n+1} > F_n$ for all $n \geq 0$. And then for $n \geq 1$, $F_{n+1} = F_n + F_{n-1} \leq F_n + F_n = 2F_n$. And $F_1 = 1 \leq 2 \cdot 1 = 2F_0$.*

(13) For any predicate P, the statement "$\forall x \in \emptyset,\ P(x)$" is true.

Solution. $\boxed{True}$. *Since there is no element in $\emptyset$, no matter what P is, this statement is vacuously true.*

(14) For any three sets A, B, C, we have $(A - B) \cup (B - C) = (A - C)$.

Solution. $\boxed{False}$. *There are numerous counterexamples, like* $A = C = \{0\}, B = \emptyset$.

(15) There exists a set with the largest cardinality.

Solution. $\boxed{False}$. *For any set* S, $\mathcal{P}(S)$ *always has greater cardinality than* S.

(16) Every function is either one-to-one or onto.

Solution. $\boxed{False}$. *One counterexample is* $f : \mathbb{Z} \to \mathbb{Z}$ *such that* $f(n) = n^2$.

(17) Let $b > 0$. The domain of the logarithm function $\log_b x$ is $\mathbb{R}_+$.

Solution. $\boxed{True}$. *For real numbers* y, b^y *could be any positive real number.*

(18) If a function has its inverse function, then it is one-to-one.

Solution. $\boxed{True}$. *The existence of inverse function is equivalent to being a one-to-one correspondence, which implies the one-to-one property.*

(19) Let R be a relation on a set A. Then for all $x \in A$, $(x, x) \in R$.

Solution. $\boxed{False}$. *This condition is the reflexive property of relations, which is not always true.*

(20) An equivalence relation on a set A always has finitely many equivalence classes.

Solution. $\boxed{False}$. *If* A *is finite, there are many counterexamples, like the empty relation* ($R = \emptyset$).

(21) In probability theory, an event is always a single outcome in the sample space.

Solution. $\boxed{False}$. *An event is a* subset *of the sample space and it may contain multiple outcomes.*

(22) The probability of an event is always nonnegative.

Solution. $\boxed{True}$. *This follows from the definition and it does not make sense to say that the probability of some event is negative.*

(23) The pigeonhole principle is an axiom.

Solution. $\boxed{False}$. *It is a theorem that could be proved, especially the version about functions.*

(24) For any real number x, we have $\lceil x \rceil < x + 1$, where $\lceil x \rceil$ is the ceiling function of x.

Solution. $\boxed{True}$. *It is true in both cases that x is an integer or not.*

2. Multiple choices: there is **exactly one** correct answer for each question.

(1) Which of the following statement forms is logically equivalent to $\sim p \to q$?

(a) $q \to p$.

(b) $p \wedge q$.

(c) $p \vee q$.

(d) $\sim q \to \sim p$.

Solution. *The answer is* $\boxed{(c)}$. *It could be verified by the truth tables.*

(2) Which of the following is NOT logically equivalent to the negation of the statement "All discrete mathematics students are athletic"?

(a) There is a discrete mathematics student who is nonathletic.

(b) There is an athletic person who is not a discrete mathematics student.

(c) Some discrete mathematics students are nonathletic.

(d) Some nonathletic people are discrete mathematics students.

Solution. *The answer is* $\boxed{(b)}$. *The negation would means that the set of discrete mathematics students and the set of nonathletic people has a nonempty intersection. All of (a), (c) and (d) exactly means this, while (b) does not.*

(3) What is the flaw of the following proof of the statement "for all integers $n \geq 0$, $5 \cdot n = 0$"?

Proof. Basis step: when $n = 0$, we have $5 \cdot 0 = 0$.

Inductive step: we apply the strong induction. Suppose $k \geq 0$ is an integer such that $5 \cdot j = 0$ for all nonnegative integers j with $0 \leq j \leq k$. We write $k+1 = i + j$, where i and j are nonnegative integers less than $k+1$. By the induction hypothesis, $5(k+1) = 5(i+j) = 5i + 5j = 0 + 0 = 0$. The inductive step is done. □

(a) The basis step is wrong.

(b) The induction hypothesis is wrongly stated.

(c) The inductive step is wrong for all $k \geq 0$.

(d) The inductive step is only wrong for $k = 0$.

Solution. *The answer is* $\boxed{(d)}$. *There is not flaw in the basis step and the description of the induction hypothesis. When* $k \geq 1$*, we can take* $i = k, j = 1$ *and the induction hypothesis applies to both* i *and* j*, so the inductive step would be* correct. *But the problem is when* $k = 0$. *One of* i, j *must be* 1*, which means the inductive hypothesis does not apply to it.*

(4) What is the correct comment for the following proof of the statement "the difference between any odd integer and any even integer is odd"?

Proof. Suppose n is any odd integer, and m is any even integer. By definition of odd, $n = 2k + 1$ where k is an integer, and by definition of even, $m = 2k$ where k is an integer. Then

$$n - m = (2k + 1) - 2k = 1.$$

But 1 is odd. Therefore, the difference between any odd integer and any even integer is odd. □

(a) The proof is correct.

(b) The proof is incorrect because the two integers are related to each other and we cannot choose two independent parameters m and n for them.

(c) The proof is incorrect because after choosing k such that $n = 2k + 1$, k is already fixed and it may not satisfy $m = 2k$.

(d) The proof is incorrect somewhere simply because the statement is false.

Solution. *The answer is* $\boxed{(c)}$. *Actually the statement is true. And we can assign* n *and* m *to the two numbers, but there is a flaw of the proof as stated in (c).*

(5) Which of the following alternative patterns of induction proofs of the statement $\forall n \in \mathbb{N}$, $P(n)$ is incorrect?

(a) Show the following statements: $P(1)$ is true; for all $n \in \mathbb{N}$, if $P(n)$ is true, then $P(2n)$ is true; for all $n \in \mathbb{N}$, if $P(n + 1)$ is true, then $P(n)$ is true.

(b) Show the following statements: $P(1)$ and $P(2)$ are true; for all $n \in \mathbb{N}$, if $P(n)$ is true, then $P(n + 2)$ is true.

(c) Show the following statements: $P(1)$ is true; for all $n \in \mathbb{N}$, if $P(n)$ is true, then $P(2n)$ is true; for all $n \in \mathbb{N}$, if $P(n)$ is true, then $P(3n)$ is true.

(d) Introduce another predicate $Q(n)$ and show the following statements: $P(1)$ is true; for all $n \in \mathbb{N}$, if $P(n)$ is true, then $Q(n)$ is true; for all $n \in \mathbb{N}$, if $Q(n)$ is true, then $P(n + 1)$ is true.

Solution. *The answer is* $\boxed{(c)}$. *In (c), since* 5 *is neither a multiple of* 2 *nor a multiple of* 3*,* $P(5)$ *cannot be justified. For (a), every integer* n *can be covered from top - find a larger integer* $m > n$ *which is a power of* 2*, so* $P(m)$ *is justified and so are* $P(m-1), P(m-2), P(m-3), \ldots$ *until* $P(n)$*. For (b), all odd positive numbers are covered from* 1 *and all even numbers are covered from* 2*. For (d), by transitivity and* $P(n) \to Q(n)$ *and* $Q(n) \to P(n+1)$ *we get* $P(n) \to P(n+1)$*, which is the inductive step of the original version of induction.*

(6) The logician Raymond Smullyan describes an island containing two types of people: knights who always tell the truth and knaves who always lie. Suppose you meet a group of four natives A, B, C, D on this island and they describe their types to you as follows:

- A says: None of us is a knight.
- B says: Exactly one of us is a knight.
- C says: Exactly two of us are knights.
- D says: Exactly three of us are knights.

How many knights are there among them?

(a) 0.

(b) 1.

(c) 2.

(d) 3.

Solution. *The answer is* $\boxed{(b)}$. *Note that any two of them are saying something contradictory. So there are at most* 1 *knight among them. The number is either* 0 *or* 1*. So either* A *or* B *is telling the truth, and one of them must be a knight. It turns out that* B *is the only knight among them.*

3. Without using truth tables, show that $p \to (q \to p)$ is a tautology.

Proof. By the negation law, $p \vee {\sim}p$ is a tautology. By the universal bound law, $(p \vee {\sim}p) \vee {\sim}q$ is a tautology. By the definition of $\to$,

$$p \to (q \to p) \equiv {\sim}p \vee (q \to p) \equiv {\sim}p \vee ({\sim}q \vee p).$$

By the commutative and associative laws, we have

$${\sim}p \vee ({\sim}q \vee p) \equiv (p \vee {\sim}p) \vee {\sim}q.$$

So it is a tautology. $\square$

4. Is the statement "all occurrences of the letter u in 'Discrete Mathematics' are lowercase" true or false? Justify your answer.

Solution. *The answer is* $\boxed{\textit{true}}$. *Apparently there is no letter u in 'Discrete Mathematics', which means the domain of the predicate is empty, then no matter what is the predicate, the statement is always true.*

5. Let S be the set of all UT students and C be the set of all UT courses. The binary predicate $R(s, c)$ means "student s registers for course c". Rephrase the statement

$$\exists s_1 \in S, \exists s_2 \in S \text{ such that } \forall c \in C, \sim (R(s_1, c) \wedge R(s_2, c)).$$

in a sentence, and write down its negation.

Solution. *The statement is*

$\boxed{\textit{There exist two UT students } s_1 \textit{ and } s_2 \textit{ such that for any UT course } c,\ s_1 \textit{ and } s_2 \textit{ do not both register for } c}$.

Its negation is

$\boxed{\textit{for any two UT students } s_1 \textit{ and } s_2 \textit{ there exists one UT course } c \textit{ such that } s_1 \textit{ and } s_2 \textit{ both register for } c}$.

6. Let a and b be positive integers. Consider the following nonempty set of positive integers

$$\{n \in \mathbb{N} \mid \exists\, u, v \in \mathbb{Z} \text{ such that } au + bv = n\}.$$

By the well-ordering principle of the integers, this set has a smallest element d. Prove that d divides a.

Proof. Consider the result of a divided by d. By the quotient-remainder theorem, there exist integers q, r with $0 \leq r \leq d - 1$ such that

$$a = qd + r.$$

Since d belongs to the set, there exist integers u, v such that

$$d = au + bv.$$

Then

$$r = a - qd = a - q(au + bv) = a(1 - qu) + b(-qv).$$

Since q, u, v are integers, so are $1 - qu$ and $-qv$. Since d is the smallest element of the set and $r < d$, r does not belong to the set. Hence $r \notin \mathbb{N}$. So $r = 0$ and d divides a. □

7. Prove that $2^n < (n + 1)!$ for all integers $n \geq 2$.

Proof. We apply induction on n. Basis step is when $n = 2$. Since $2^2 = 4 < 6 = (2 + 1)!$, the statement is true when $n = 2$. As for the inductive step, suppose integer $k \geq 2$ such that the statement is true for $n = k$. The inductive hypothesis is

$$2^k < (k + 1)!.$$

Now we consider the case when $n = k + 1$. We have

$$2^{k+1} = 2 \cdot 2^k < 2 \cdot (k+1)! < (k+2) \cdot (k+1)! = (k+2)!.$$

So the inductive step is done. $\square$

8. Let $x > 0$ and s_n be a sequence such that $s_1 = x + \frac{1}{x}$, and $s_{n+1} = s_n^2 - 2$ for all integers $n \geq 1$. Find a closed formula for s_n. Justify your answer.

Solution. *We claim that for integers $n \geq 1$,*

$$\boxed{s_n = x^{2^{n-1}} + x^{-2^{n-1}}}.$$

We prove the claim by induction on n. Basis step is the case when $n = 1$. Then $2^{n-1} = 1$, and the claim is true. As for the inductive step, suppose integer $k \geq 1$ such that the claim is true for $n = k$. The inductive hypothesis is

$$s_k = x^{2^{k-1}} + x^{-2^{k-1}}.$$

Now consider the case when $n = k + 1$. We have

$$s_{k+1} = s_k^2 - 2 = \left[x^{2^{k-1}} + x^{-2^{k-1}}\right]^2 - 2 = x^{2^k} + x^{-2^k} + 2 - 2 = x^{2^k} + x^{-2^k}.$$

So the inductive step is done.

9. Let F_n be the n-th Fibonacci number.
(1) Show that $F_{n+3} = 2F_{n+1} + F_n$ for all integers $n \geq 0$.
(2) Show that $F_{n+4} = 3F_{n+1} + 2F_n$ for all integers $n \geq 0$.
(3) Show that $F_{n+m} = F_m F_{n+1} + F_{m-1} F_n$ for all integers $n \geq 0$ and $m \geq 1$.

Proof. (1) $F_{n+3} = F_{n+2} + F_{n+1} = (F_{n+1} + F_n) + F_{n+1} = 2F_{n+1} + F_n$.
(2) $F_{n+4} = F_{n+3} + F_{n+2} = (2F_{n+1} + F_n) + (F_{n+1} + F_n) = 3F_{n+1} + 2F_n$.
(3) We apply strong induction on m. Basis step is the case when $m = 1, 2$. When $m = 1$, note that $F_0 = 0, F_1 = 1$, so the statement becomes

$$F_{n+1} = 1 \cdot F_{n+1} + 0 \cdot F_n,$$

which is trivially true. When $m = 2$, note that $F_2 = 1$, the statement becomes

$$F_{n+2} = 1 \cdot F_{n+1} + 1 \cdot F_n,$$

which is the recursive relation of Fibonacci numbers. Basis step is done.
As for the inductive step, suppose $k \geq 2$ is an integer such that the statement is true for all integers $n \geq 0$ and $1 \leq m \leq k$. Now we consider the case when $m = k + 1$. Since $k \geq 2$, we have

$$F_{k+1} = F_k + F_{k-1}, F_k = F_{k-1} + F_{k-2}.$$

For all integers $n \geq 0$, we have

$$F_{n+k+1} = F_{n+k} + F_{n+k-1}.$$

By the inductive hypothesis when $m = k$, we have

$$F_{n+k} = F_k F_{n+1} + F_{k-1} F_n.$$

Note that $1 \leq k-1 \leq k$, by the inductive hypothesis when $m = k-1$, we have

$$F_{n+k-1} = F_{k-1} F_{n+1} + F_{k-2} F_n.$$

Hence

$$\begin{aligned} F_{n+k+1} &= (F_k F_{n+1} + F_{k-1} F_n) + (F_{k-1} F_{n+1} + F_{k-2} F_n) \\ &= (F_k + F_{k-1}) F_{n+1} + (F_{k-1} + F_{k-2}) F_n \\ &= F_{k+1} F_{n+1} + F_k F_n. \end{aligned}$$

The inductive step is done.

$\square$

10. Prove the following distributive law of sets: for any sets A, B, C,

$$A \times (B \cap C) = (A \times B) \cap (A \times C).$$

Proof. First, we show that $A \times (B \cap C) \subseteq (A \times B) \cap (A \times C)$. For any $(x, y) \in A \times (B \cap C)$, by the definition of Cartesian product, $x \in A$ and $y \in B \cap C$. Hence $y \in B$ and $y \in C$. Then $(x, y) \in A \times B$, and $(x, y) \in A \times C$. So $(x, y) \in (A \times B) \cap (A \times C)$.
Second, we show that $(A \times B) \cap (A \times C) \subseteq A \times (B \cap C)$. For any pair $(x, y) \in (A \times B) \cap (A \times C)$, we know that $(x, y) \in A \times B$ and $(x, y) \in A \times C$. By the definition of Cartesian product, $(x, y) \in A \times B$ implies $x \in A$ and $y \in B$; $(x, y) \in A \times C$ implies $x \in A$ and $y \in C$. Hence $y \in B \cap C$. Then $(x, y) \in A \times (B \cap C)$. In summary, $A \times (B \cap C) = (A \times B) \cap (A \times C)$. $\square$

11. Construct a one-to-one function $f : \mathbb{N} \to \mathbb{N}$ such that the range of f does not contain any prime number. Justify your answer.

Solution. *There are many correct answers. Just make sure that the range of f consider composite numbers or 1 only. Note that even numbers at least 4 are all composite, one example would be*

$$\boxed{f(n) = 2n + 2 \; \forall \, n \in \mathbb{N}}.$$

12. Consider the binary function $g : \mathbb{Q} \times \mathbb{Q} \to \mathbb{R}$ defined by $g(r, s) = r + \sqrt{3}s$. Is g one-to-one? Is g onto? Justify your answer.

Solution. *g $\boxed{\text{is}}$ one-to-one. g $\boxed{\text{is not}}$ onto. Suppose $r_1, s_1, r_2, s_2 \in \mathbb{Q}$ such that $g(r_1, s_1) = g(r_2, s_2)$, then*

$$r_1 + \sqrt{3}s_1 = r_2 + \sqrt{3}s_2$$

and

$$r_1 - r_2 = \sqrt{3} \cdot (s_2 - s_1).$$

Suppose $s_2 - s_1 \neq 0$, then

$$\sqrt{3} = \frac{r_1 - r_2}{s_2 - s_1}.$$

Since $r_1, s_1, r_2, s_2 \in \mathbb{Q}$, so are $r_1 - r_2, s_2 - s_1$ and their ratio, which is a contradiction to the fact that $\sqrt{3}$ is irrational! Hence $s_2 - s_1 = 0$ and thus $r_1 - r_2 = 0$. So $(r_1, s_1) = (r_2, s_2)$, which proves the one-to-one property of g.

Suppose g is onto as well, then g is a one-to-one correspondence between $\mathbb{Q} \times \mathbb{Q}$ and $\mathbb{R}$. By definition, these two sets must have the same cardinality. However, $\mathbb{Q}$ is countable, so is $\mathbb{Q} \times \mathbb{Q}$. While $\mathbb{R}$ is uncountable, a contradiction! Hence g is not onto.

13. Let C be the set of all points in a circle with radius 1. Let R be a relation on C such that $(x, y) \in R$ if and only if the distance between x and y is less than 0.5. Is R an equivalence relation? Justify your answer.

Solution. *The answer is $\boxed{\text{no}}$. R does satisfy reflexive and symmetric properties. However it does not satisfy transitivity. Let O be the center of the circle, A is an arbitrary point on the circle, and M be the midpoint of the line segment AO. Then the distance between M, A and M, O are both 0.25, while the distance between A and o is 0.5. By definition, we have*

$$(O, M) \in R, (M, A) \in R, (O, A) \notin R.$$

So R is not transitive and it is not an equivalence relation.

14. Let R be the equivalence relation defined on $\mathbb{Z}$ such that $(x, y) \in R$ if and only if $7 \mid (x^2 - y^2)$. Find the equivalence classes of R.

Solution. *Note that if $7 \mid (x - y)$, then $7 \mid \left(x^2 - y^2\right)$, so we only need to consider integers $0, 1, 2, 3, 4, 5, 6$. We have the following table:*

n	0	1	2	3	4	5	6
n^2	0	1	4	9	16	25	36
$n^2 \mod 7$	0	1	4	2	2	4	1

So there are 4 equivalence classes:

$$\boxed{\{7k \mid k \in \mathbb{Z}\}, \{7k \pm 1 \mid k \in \mathbb{Z}\}, \{7k \pm 2 \mid k \in \mathbb{Z}\}, \{7k \pm 3 \mid k \in \mathbb{Z}\}}.$$

15. What is the unit digit of 3^{361} (the number of possible configurations on a *Go* board)?

Not in the range of the final exam.

16. If you roll 3 normal dice together, what is the probability that at least two of them have equal outcomes?

Solution. *The sample space has* $6^3 = 216$ *outcomes. To compute the cardinality of the event "at least two dice have equal outcomes", the easiest way is to apply the difference rule and consider its opposite: the event "all three dice have different outcomes". Each case is simply a* 3*-permutation of the six numbers, so the cardinality of this alternative case is* $P(6,3) = 6 \cdot 5 \cdot 4 = 120$*. So the cardinality of the original event is* $216 - 120 = 96$*. The answer is*

$$\boxed{\frac{96}{216} = \frac{4}{9}} \approx 44.4\%.$$

17. *Roulette* is one of the simplest casino game to play: there are 38 equally likely outcomes labeled on a wheel. 18 of them are red, 18 of them are black, and the remaining two are green. One may bet on either red or black. If the outcome matches one's bet, one wins the same amount of the bet, otherwise one losses the bet. What is the probability that one wins a single bet? Justify your answer.
(Warning: after learning probability theory, you will understand that even the games are fair, you will still lose money to the casino if you bet sufficiently many times in any casino games)

Solution. *The sample space has* 38 *equally likely outcomes. Both events "the outcome is red" and "the outcome is black" have cardinality* 18*, so the probability of winning is always*

$$\boxed{\frac{18}{38} = \frac{9}{19}} \approx 47.3\%.$$

18. An interesting use of the inclusion/exclusion rule is to check survey numbers for consistency. For example, suppose a public opinion polltaker reports that out of a national sample of $1,200$ adults, 675 are married, 682 are from 20 to 30 years old, 684 are female, 195 are married and are from 20 to 30 years old, 467 are married females, 318 are females from 20 to 30 years old, and 165 are married females from 20 to 30 years old. Are the polltakers figures consistent? Could they have occurred as a result of an actual sample survey?

Solution. *Suppose the figures are correct. By the inclusion/exclusion rule on three sets, the total number of people who are female or married or from* 20 *to* 30 *years old is*

$$675 + 682 + 684 - 195 - 467 - 318 + 165 = 1,226,$$

which is greater than the total number of surveyees. So the figures are $\boxed{\textit{not}}$ *consistent they can not occur as a result of an actual sample survey.*

19. Prove that if you choose 7 integers between 2 and 13 inclusive, there exist two chosen numbers such that neither divides the other.

Proof. We partition the integers into 6 subsets:

$$\{2,3\},\{4,5\},\{6,7\},\{8,9\},\{10,11\},\{12,13\}.$$

Note that in each subset, neither of the two numbers divides the other. By the pigeonhole principle, since you choose 7 numbers and $7 > 6$, there exists a subset with at least 2 chosen numbers (and exactly 2). This pair justifies the statement. □

20. Prove that for all positive integers n,

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} = 0.$$

Proof. We plug-in $a = -1, b = 1$ in the binomial theorem. The left-hand side becomes $(-1+1)^n = 0$. The right hand side becomes

$$\sum_{i=0}^{n} \binom{n}{i} (-1)^i 1^{n-i} = \sum_{i=0}^{n} (-1)^i \binom{n}{i}.$$

So the statement follows from the binomial theorem. □