Math 325K Fall 2018
Practice problem set

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This problem set is to help you prepare for the final exam. Most final exam problems would be of similar style as some problems in this set.

1. True/False: each of the following arguments is either true or false.

(1) If the premise of a conditional statement is false, then the statement itself is false too.

(2) Let A, B be two sets. The Cartesian products $A \times B$ and $B \times A$ are NOT always the same.

(3) Let p, q be two statements. If $p \vee q$ is false, then $p \wedge q$ is false too.

(4) The inverse and the converse of the same conditional statement are logically equivalent.

(5) To disprove a universal statement, one counterexample is enough.

(6) In multi-quantified statements, the statement remains the same if we change the order of the quantifiers.

(7) Every positive integer is either prime or composite.

(8) If the product of two integers a, b is even, then at least one of them is also even.

(9) The difference of any two rational numbers is still rational.

(10) Suppose one justifies the statement $P(a)$ in the basis step of a proof by induction, then $a \geq 0$.

(11) $\sum_{i=1}^{100} i^2 = \sum_{j=1}^{100} j^2$.

(12) Let $\{F_n\}$ be the Fibonacci sequence. Then $F_{n+1} \leq 2F_n$ for all positive integers n.

(13) For any predicate P, the statement "$\forall x \in \emptyset,\ P(x)$" is true.

(14) For any three sets A, B, C, we have $(A - B) \cup (B - C) = (A - C)$.

(15) There exists a set with the largest cardinality.

(16) Every function is either one-to-one or onto.

(17) Let $b > 0$. The domain of the logarithm function $\log_b x$ is $\mathbb{R}_+$.

(18) If a function has its inverse function, then it is one-to-one.

(19) Let R be a relation on a set A. Then for all $x \in A$, $(x, x) \in R$.

(20) An equivalence relation on a set A always has finitely many equivalence classes.

(21) In probability theory, an event is always a single outcome in the sample space.

(22) The probability of an event is always nonnegative.

(23) The pigeonhole principle is an axiom.

(24) For any real number x, we have $\lceil x \rceil < x + 1$, where $\lceil x \rceil$ is the ceiling function of x.

2. Multiple choices: there is **exactly one** correct answer for each question.

(1) Which of the following statement forms is logically equivalent to $\sim p \to q$?

(a) $q \to p$.

(b) $p \wedge q$.

(c) $p \vee q$.

(d) $\sim q \to \sim p$.

(2) Which of the following is NOT logically equivalent to the negation of the statement "All discrete mathematics students are athletic"?

(a) There is a discrete mathematics student who is nonathletic.

(b) There is an athletic person who is not a discrete mathematics student.

(c) Some discrete mathematics students are nonathletic.

(d) Some nonathletic people are discrete mathematics students.

(3) What is the flaw of the following proof of the statement "for all integers $n \geq 0$, $5 \cdot n = 0$"?

Proof. Basis step: when $n = 0$, we have $5 \cdot 0 = 0$.

Inductive step: we apply the strong induction. Suppose $k \geq 0$ is an integer such that $5 \cdot j = 0$ for all nonnegative integers j with $0 \leq j \leq k$. We write $k+1 = i + j$, where i and j are nonnegative integers less than $k+1$. By the induction hypothesis, $5(k+1) = 5(i+j) = 5i + 5j = 0 + 0 = 0$. The inductive step is done. $\square$

(a) The basis step is wrong.

(b) The induction hypothesis is wrongly stated.

(c) The inductive step is wrong for all $k \geq 0$.

(d) The inductive step is only wrong for $k = 0$.

(4) What is the correct comment for the following proof of the statement "the difference between any odd integer and any even integer is odd"?

Proof. Suppose n is any odd integer, and m is any even integer. By definition of odd, $n = 2k + 1$ where k is an integer, and by definition of even, $m = 2k$ where k is an integer. Then

$$n - m = (2k + 1) - 2k = 1.$$

But 1 is odd. Therefore, the difference between any odd integer and any even integer is odd. □

(a) The proof is correct.

(b) The proof is incorrect because the two integers are related to each other and we cannot choose two independent parameters m and n for them.

(c) The proof is incorrect because after choosing k such that $n = 2k + 1$, k is already fixed and it may not satisfy $m = 2k$.

(d) The proof is incorrect somewhere simply because the statement is false.

(5) Which of the following alternative patterns of induction proofs of the statement $\forall n \in \mathbb{N}$, $P(n)$ is incorrect?

(a) Show the following statements: $P(1)$ is true; for all $n \in \mathbb{N}$, if $P(n)$ is true, then $P(2n)$ is true; for all $n \in \mathbb{N}$, if $P(n + 1)$ is true, then $P(n)$ is true.

(b) Show the following statements: $P(1)$ and $P(2)$ are true; for all $n \in \mathbb{N}$, if $P(n)$ is true, then $P(n + 2)$ is true.

(c) Show the following statements: $P(1)$ is true; for all $n \in \mathbb{N}$, if $P(n)$ is true, then $P(2n)$ is true; for all $n \in \mathbb{N}$, if $P(n)$ is true, then $P(3n)$ is true.

(d) Introduce another predicate $Q(n)$ and show the following statements: $P(1)$ is true; for all $n \in \mathbb{N}$, if $P(n)$ is true, then $Q(n)$ is true; for all $n \in \mathbb{N}$, if $Q(n)$ is true, then $P(n+1)$ is true.

(6) The logician Raymond Smullyan describes an island containing two types of people: knights who always tell the truth and knaves who always lie. Suppose you meet a group of four natives A, B, C, D on this island and they describe their types to you as follows:

- A says: None of us is a knight.
- B says: Exactly one of us is a knight.
- C says: Exactly two of us are knights.
- D says: Exactly three of us are knights.

How many knights are there among them?

(a) 0.

(b) 1.

(c) 2.

(d) 3.

3. Without using truth tables, show that $p \to (q \to p)$ is a tautology.

4. Is the statement "all occurrences of the letter u in 'Discrete Mathematics' are lowercase" true or false? Justify your answer.

5. Let S be the set of all UT students and C be the set of all UT courses. The binary predicate $R(s, c)$ means "student s registers in course c". Rephrase the statement

$$\exists s_1 \in S, \exists s_2 \in S \text{ such that } \forall c \in C, \sim (R(s_1, c) \wedge R(s_2, c)) \,.$$

in a sentence, and write down its negation.

6. Let a and b be positive integers. Consider the following nonempty set of positive integers

$$\{n \in \mathbb{N} \mid \exists\, u, v \in \mathbb{Z} \text{ such that } au + bv = n\}.$$

By the well-ordering principle of the integers, this set has a smallest element d. Prove that d divides a.

7. Prove that $2^n < (n+1)!$ for all integers $n \geq 2$.

8. Let $x > 0$ and s_n be a sequence such that $s_1 = x + \frac{1}{x}$, and $s_{n+1} = s_n^2 - 2$ for all integers $n \geq 1$. Find a closed formula for s_n. Justify your answer.

9. Let F_n be the n-th Fibonacci number.
(1) Show that $F_{n+3} = 2F_{n+1} + F_n$ for all integers $n \geq 0$.
(2) Show that $F_{n+4} = 3F_{n+1} + 2F_n$ for all integers $n \geq 0$.
(3) Show that $F_{n+m} = F_m F_{n+1} + F_{m-1} F_n$ for all integers $n \geq 0$ and $m \geq 1$.

10. Prove the following distributive law of sets: for any sets A, B, C,

$$A \times (B \cap C) = (A \times B) \cap (A \times C).$$

11. Construct a one-to-one function $f : \mathbb{N} \to \mathbb{N}$ such that the range of f does not contain any prime number. Justify your answer.

12. Consider the binary function $g : \mathbb{Q} \times \mathbb{Q} \to \mathbb{R}$ defined by $g(r, s) = r + \sqrt{3}s$. Is g one-to-one? Is g onto? Justify your answer.

13. Let C be the set of all points in a circle with radius 1. Let R be a relation on C such that $(x, y) \in R$ if and only if the distance between x and y is less than 0.5. Is R an equivalence relation? Justify your answer.

14. Let R be the equivalence relation defined on $\mathbb{Z}$ such that $(x, y) \in R$ if and only if $7 \mid (x^2 - y^2)$. Find the equivalence classes of R.

15. What is the unit digit of 3^{361} (the number of possible configurations on a *Go* board)?

16. If you roll 3 normal dice together, what is the probability that at least two of them have equal outcome?

17. *Roulette* is one of the simplest casino game to play: there are 38 equally likely outcomes labeled on a wheel. 18 of them are red, 18 of them are black, and the remaining two are green. One may bet on either red or black. If the outcome matches one's bet, one wins the same amount of the bet, otherwise one losses the bet. What is the probability that one wins a single bet? Justify your answer.
(Warning: after learning probability theory, you will understand that even the games are fair, you will still lose money to the casino if you bet sufficiently many times in any casino games)

18. An interesting use of the inclusion/exclusion rule is to check survey numbers for consistency. For example, suppose a public opinion polltaker reports that out of a national sample of $1,200$ adults, 675 are married, 682 are from 20 to 30 years old, 684 are female, 195 are married and are from 20 to 30 years old, 467 are married females, 318 are females from 20 to 30 years old, and 165 are married females from 20 to 30 years old. Are the polltakers figures consistent? Could they have occurred as a result of an actual sample survey?

19. Prove that if you choose 7 integers between 2 and 13 inclusive, there exist two chosen numbers such that neither divides the other.

20. Prove that for all positive integers n,

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} = 0.$$