Math 325K Fall 2018
Final Exam Solutions

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1. (16 pts) True/False: each of the following arguments is either true or false and please mark your choice. You get 2 pts for each correct choice, 1 pt for **NOT** answering each question, and 0 pt for each incorrect/multiple choice. You **do not** need to justify your answer.

(1) Any conditional statement and its contraposition are logically equivalent.

Solution. $\boxed{True}$. *One can verify by truth table.*

(2) To justify a universal statement, one eligible example is always enough.

Solution. $\boxed{False}$. *We need to show that all elements in the domain lead to eligible examples.*

(3) For every positive integer n, if n^2 is even, then n must be even too.

Solution. $\boxed{True}$. *Because if n is odd, then there exists an integer k such that $n = 2k+1$. So $n^2 = (2k+1)^2 = 4k^2+4k+1$ is odd too, a contradiction. Hence n must be even.*

(4) Let A, B be subsets of the universal set U. Then $(A \cup B)^c = A^c \cup B^c$.

Solution. $\boxed{False}$. *De Morgan's Laws tell us that $(A \cup B)^c = A^c \cap B^c$. One explicit counterexample is*

$$U = \{1, 2\}, A = \{1\}, B = \{2\}.$$

Then

$$A \cup B = U, (A \cup B)^c = \emptyset.$$

But

$$A^c \cup B^c = U.$$

(5) A sequence cannot be both arithmetic and geometric.

Solution. $\boxed{False}$. *One counterexample would be the constant sequence* $a_n = 1$*, which is an arithmetic sequence with common difference zero and a geometric sequence with common ratio* 1*.*

(6) Let R be a relation defined on a set A. If R is symmetric and transitive, then it must be reflexive too.

Solution. $\boxed{False}$. *See Exercise 39 of Section 8.3.*

(7) If infinitely many objects are put into finitely many sets, there must be a set with infinitely many objects.

Solution. $\boxed{True}$. *This is a generalized version of the pigeonhole principle. Suppose all set only contain finitely many objects, then their union contains finitely many objects too, a contradiction to the fact that infinitely many objects are put into the union.*

(8) If one rolls a normal 6-side die once, the probability of the event "the outcome is a prime number" is less than the probability of the event "the outcome is an even number".

Solution. $\boxed{False}$. *The sample space is* $\{1, 2, 3, 4, 5, 6\}$*. The first event is the subset* $\{2, 3, 5\}$ *and the second event is the subset* $\{2, 4, 6\}$*, so their cardinalities are equal, so are their probabilities.*

2. (12 pts) Multiple choices: there is **exactly one** correct answer for each question. You get 4 pts for each correct choice, 1 pt for **NOT** answering each question, and 0 pt for each incorrect/multiple choice. You **do not** need to justify your answer.

(1) Here is a proof by mathematical induction of the statement "for all $n \in \mathbb{N}$, if $x, y \in \mathbb{N}$ such that $\max(x, y) = n$, then $x = y$".

Proof. We apply induction on n. Basis step is the case when $n = 1$. If $\max(x, y) = 1$, then both x and y are at most 1. Since they are positive integers, both must be 1 and they are equal. For the inductive step, suppose $k \in \mathbb{N}$ such that the statement is true for $n = k$, consider the case when $n = k + 1$. For $x, y \in \mathbb{N}$, if $\max(x, y) = k + 1$, then $\max(x - 1, y - 1) = k$. By the inductive hypothesis, we have that $x - 1 = y - 1$. Hence $x = y$ and the inductive step is done. □

What is the correct comment about the statement and the proof?

(a) The statement is correct and the proof is correct too.

(b) The statement is correct, but the proof is flawed because the basis step is incorrect.

(c) The statement is false, and the inductive step is false for $k = 1$ only.

(d) The statement is false, and the inductive step is false for all $k \geq 1$.

Solution. *The answer is* $\boxed{(d)}$. *First, the statement is obviously false (one counterexample could be* $n = 2, x = 1, y = 2$*). So we need to decide between (c) and (d). The flaw is the step when applying the inductive hypothesis. To apply it to* $x - 1, y - 1$*, we need to make sure that both* $x - 1$ *and* $y - 1$ *belong to* $\mathbb{N}$*. The only constraint on* x *and* y *is that* $x, y \in \mathbb{N}$*. So if* $x = 1$, $x - 1 = 0 \notin \mathbb{N}$*! And this is indeed possible because we can always take* $x = 1, y = k + 1$*. Hence the inductive step is flawed for all* $k \geq 1$*.*

(2) Let $P(n)$ be an unary predicate. Suppose our task is to justify the statement "$\forall n \in \mathbb{N}, P(n)$", and we already justified $P(1)$ and "$\forall n \in \mathbb{N}, P(n) \to P(n + 3)$". Which of the following statements is enough for our task if we can also justify it?

(a) $\forall n \in \mathbb{N},\ P(n + 1) \to P(n)$.

(b) $\forall n \in \mathbb{N},\ P(n) \to P(2n)$.

(c) $\forall n \in \mathbb{N},\ P(n) \to P(n + 2)$.

(d) $P(2)$.

Solution. *The answer is* $\boxed{(a)}$. *Suppose we can justify (a). Then by transitivity, for each* $n \in \mathbb{N}$*, we have*

$$P(n) \to P(n + 3) \to P(n + 2) \to P(n + 1),$$

which is exactly the inductive step in the ordinary version of induction.

For (b), note that for any $n \in N$*, neither* $n + 3$ *nor* $2n$ *can be* 3*, so we still cannot justify* $P(3)$*. For (c), note that for any* $n \in N$*, neither* $n + 3$ *nor* $n + 2$ *can be* 2*, so we still cannot justify* $P(2)$*. For (d), note that for any* $n \in N$*,* $n+3$ *cannot be* 3*. So we cannot justify* $P(3)$*.*

(3) The logician Raymond Smullyan describes an island containing two types of people: knights who always tell the truth and knaves who always lie. Suppose you meet a group of three natives A, B, C on this island and they describe their identities to you as follows:

- A says: B is a knight.
- B says: I am the only knight among us.
- C says: There is at least one knave among us.

How many knights are there among them?

(a) 0.

(b) 1.

(c) 2.

(d) 3.

Solution. *The answer is* $\boxed{(b)}$. *The breakthrough is the identity of B. Suppose B is a knight, then B is telling the truth and A would be a knave, but since B is a knight, A is actually telling the truth, a contradiction! So B must be a knave. Then A is telling a lie and C is telling the truth. So it turns out that C is the only knight among them.*

3. (4 pts)
Show that the statement form $(p \to q) \vee (q \to p)$ is a tautology.

Proof. Note that

$$p \to q \equiv \sim p \vee q$$

and

$$q \to p \equiv \sim q \vee p.$$

The original statement form is logically equivalent to the disjunction of $p, q, \sim p, \sim q$. By the negation law, $p \vee \sim p$ is a tautology. By the universal bound law, $(p \vee \sim p) \vee (\sim q \vee q)$ is also a tautology. □

Remark 1. *One can also prove it by a truth table.*

4. (5 pts)
Let $A = \{1, 2, 3, 4, 5\}$ and the binary predicate $T(x, y)$ be "$x+y$ is a multiple of 3". Rephrase the following statement

$$\forall x \in A\, \forall y \in A\, \forall z \in A, (T(x, y) \wedge T(y, z)) \to T(x, z)$$

in an English sentence. Is it true or false? Justify your answer.

Solution. *The statement is "for any elements x, y, z in the set A, if $x+y$ is a multiple of 3 and $y+z$ is a multiple of 3, then $x+z$ is a multiple of 3".*
The statement is $\boxed{false}$. *One counterexample would be*

$$x = 1, y = 2, z = 4.$$

Then $x+y=3, y+z=6$ are both multiples of 3, while $x+z=5$ is not.

5. (4 pts)
Show that for every positive integer n, we have

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}.$$

Proof. We apply induction on n. The basis step is the case when $n = 1$. We have

$$\sum_{i=1}^{1} i^3 = 1^3 = 1, \frac{n^2(n+1)^2}{4} = \frac{1^2 \cdot 2^2}{4} = 1.$$

For the inductive step, suppose $m \in \mathbb{N}$ such that the statement is true for $n = m$. Then

$$\sum_{i=1}^{m} i^3 = \frac{m^2(m+1)^2}{4}.$$

Now we consider the case when $n = m+1$. We have

$$\begin{aligned}
\sum_{i=1}^{m+1} i^3 &= \sum_{i=1}^{m} i^3 + (m+1)^3 \\
&= \frac{m^2(m+1)^2}{4} + (m+1)^3 \\
&= \frac{(m+1)^2}{4} \cdot \left[m^2 + 4(m+1)\right] \\
&= \frac{(m+1)^2}{4} \cdot (m^2 + 4m + 4) \\
&= \frac{(m+1)^2}{4} \cdot (m+2)^2 \\
&= \frac{(m+1)^2(m+2)^2}{4}.
\end{aligned}$$

So the statement is also true for $n = m+1$. The inductive step is done. □

6. (4 pts)
Show that for every positive integer n, we have $3^n > n^2 + 1$.

Proof. We apply induction on n. The basis step is the case when $n = 1$. We have

$$3^1 = 3 > 2 = 1^2 + 1.$$

For the inductive step, suppose $m \in \mathbb{N}$ such that the statement is true for $n = m$. Then

$$3^m > m^2 + 1.$$

Now we consider the case when $n = m + 1$. We have

$$3^{m+1} = 3{\cdot}3^m > 3{\cdot}(m^2+1) = 3m^2+3 > m^2+2m^2+2 \geq m^2+2m+2 = (m+1)^2+1.$$

So the statement is also true for $n = m + 1$. The inductive step is done. □

7. (5 pts)

Let $\{F_n\}_{n\geq 0}$ be the Fibonacci sequence. Show that for any integer $n \geq 0$, the terms F_n and F_{n+1} do not have a common divisor greater than 1 (in other words, for any integer $d > 1$, F_n and F_{n+1} cannot both be multiples of d).

Proof. We apply induction on n. The basis step is the cases when $n = 0$. Since $F_0 = 0, F_1 = 1$, F_0 and F_1 do not have a common divisor greater than 1. For the inductive step, suppose integer $m \geq 0$ such that the statement is true for $n = m$. Then F_m and F_{m+1} do not have a common divisor greater than 1. Now we consider the case when $n = m + 1$. Suppose d is a common divisor of F_{m+1} and $F_{(m+1)+1} = F_{m+2}$. By the recursive relation of Fibonacci numbers,

$$F_{m+2} = F_{m+1} + F_m.$$

Then

$$d \mid F_{m+1}, d \mid (F_{m+1} + F_m).$$

So $d \mid F_m$ and d is a common divisor of F_m and F_{m+1}. By the inductive hypothesis, d cannot be greater than 1. Hence F_{m+1} and F_{m+2} do not have a common divisor greater than 1 neither. The inductive step is done. □

8. (5 pts)

Let A, B, C, D be sets.

(1) (2 pts) Show that if $A \subseteq B$ and $C \subseteq D$, then $(A \cap C) \subseteq (B \cap D)$.

(2) (3 pts) The symbol $X \subsetneq Y$ means that X is a proper subset of Y. If $A \subsetneq B$ and $C \subsetneq D$, is it always true that $(A \cap C) \subsetneq (B \cap D)$? If yes, justify it; if no, present a counterexample.

Solution. *(1) For any $x \in A \cap C$, x must belong to both A and C. Since $x \in A$ and $A \subseteq C$, $x \in C$; since $x \in B$ and $B \subseteq D$, $x \in D$. Hence $x \in B \cap D$. By the element method, $(A \cap C) \subseteq (B \cap D)$.*
(2) The answer is $\boxed{\textit{no}}$*. One counterexample would be*

$$A = C = \emptyset, B = \{1\}, D = \{2\}.$$

Then $A \cap C = B \cap D = \emptyset$, *so* $A \cap C$ *is not a proper subset of* $B \cap D$.

9. (5 pts)

Suppose $f : \mathbb{N} \to \mathbb{N}$ is a function such that $f(f(x)) = x$ for all $x \in \mathbb{N}$. Show that f must be a one-to-one correspondence.

Proof. First we show that f is one-to-one. Suppose $x, y \in \mathbb{N}$ such that $f(x) = f(y)$. Then

$$x = f(f(x)) = f(f(y)) = y.$$

Hence f is one-to-one.

Second we show that f is onto. For any $x \in \mathbb{N}$, by definition of f, $f(x)$ is also a positive integer. In addition, f maps $f(x)$ to x, so x belongs to the range of f. Hence the range of f is $\mathbb{N}$, the same as the co-domain of f and f is onto.

Finally, since f is both one-to-one and onto, by definition, f is a one-to-one correspondence. □

10. (4 pts)

A major airline in the U.S. has a list of its scheduled direct flights, for example AUS to DFW. This list defines a relation R on the set A of all airports that appear on this list as follows: for any two airports x and y in A, $(x, y) \in R$ if and only if there is a direct flight from x to y in the list. Based on your common sense, could R be an equivalence relation? Justify your answer.

Solution. *The answer is* $\boxed{no}$. *R could be symmetric if all flight routes are round-trip. However, R cannot be reflexive at all because no actual flight would have the same starting point and destination. Hence R cannot be an equivalence relation.*

Remark 2. *Note that the question is about an existential statement: whether there exists a list such that R is an equivalence relation. In fact, R could be transitive. For example if the list only contains the following direct flights:*

$$AUS \to DFW, DFW \to IAH, AUS \to IAH.$$

As a result, it is not correct to claim that R must not be transitive. Nonetheless, R is unlikely to be transitive neither if there is a round-trip route. For example, AUS-DFW and DFW-AUS and transitivity would imply a route AUS-AUS, which does not make sense.

11. (6 pts)

All of us have our own UT EIDs. The current pattern of a UT EID is either 3 lowercase letters followed by an integer from 0 to $9,999$, or 2 lowercase letters followed by an integer from 0 to $99,999$. Find the formula of the total number of eligible UT EIDs of current pattern. Justify your answer.

Solution. *By the addition rule, the answer would be the sum of the numbers of the two patterns. For the first pattern, we choose* 3 *lowercase letters and then choose an integer from* 0 *to* 9,999*. Since there are* 26 *lowercase letters, by the multiplication rule, the total number of choices in this case is*

$$26^3 \cdot 10,000.$$

Similarly, for the second pattern, we choose 2 *lowercase letters and then choose an integer from* 0 *to* 99,999*. By the multiplication rule, the total number of choices in this case is*

$$26^2 \cdot 100,000.$$

So the answer is

$$\boxed{26^3 \cdot 10,000 + 26^2 \cdot 100,000 = 26^2 \cdot 10,000 \cdot 36} = 243,360,000.$$

Remark 3. *There are UT EIDs of other patterns in older times, for example one of my colleagues' is a sequence pure of lowercase letters.*

12. (6 pts)

On a weekday afternoon you wait for line 640 or 642 at a bus stop. According to the timetable, there are line 640 buses leaving at 1:05, 1:14 and 1:23 (every 9 minutes); and there are line 642 buses leaving at 1:06, 1:18 and 1:30 (every 12 minutes). You arrive x minutes past 1 o'clock, where x is a random integer from 1 to 30 with equal likelihood. You will take the next bus of either line that you can catch. For example, if you arrive at 1:10, you will catch the 640 bus at 1:14; if you arrive at 1:06, you can still catch the 642 bus. What is the probability that you end up catching a line 640 bus? Justify your answer.

Solution. *The sample space has cardinality* 30*. In order to compute the probability, we need to count the number of outcomes (arrival times) that you end up catching a line 640 bus. The outcomes that you end up catching the 1:05 line 640 bus are*

$$x = 1, 2, 3, 4, 5.$$

The outcomes that you end up catching the 1:14 line 640 bus are

$$x = 7, 8, 9, 10, 11, 12, 13, 14.$$

And the outcomes that you end up catching the 1:23 line 640 bus are

$$x = 19, 20, 21, 22, 23.$$

So the cardinality of this event is $5 + 8 + 5 = 18$*, and the answer is*

$$\boxed{\frac{18}{30} = 60\%}.$$

Remark 4. *Suppose line 642 buses also come every 9 minutes, say at 1:06, 1:15, and 1:24, then your first guess of the answer would be close to 50%, while the actual answer would shock you.*

13. (4 pts)

How many cards at least must one pick from a standard 52-card deck to be sure of getting at least 2 cards in the same suit? Justify your answer.

Solution. *On one hand, Note that there are 4 suits in a standard 52-card deck. If one pick 4 cards, it is possible that the picked cards are the 4 aces of all suits, then this combination of cards is not eligible because there are no 2 cards in the same suit. So 4 is not enough.*

On the other hand, if one picks 5 cards. Since $5 > 4$, by the pigeonhole principle, there exists a suit which contains at least 2 picked cards. Hence 5 works. And the answer is $\boxed{5}$.

Extra Problem I (5 pts)

The "24 game" is an arithmetic card game. In each round, one deals a hand of four cards from a standard 52-card deck and takes the four corresponding numbers (for example $\spadesuit Q$ corresponds to 12, and $\diamondsuit 6$ corresponds to 6) then does arithmetic operations to get 24. The total number of possible hands is the total number of unordered selections of 4 integers from 1 to 13 where equal numbers are allowed (for example $1, 1, 7, 11$). Find this number and justify your answer. (Hint: the solution is not related to the number 24)

Solution. *Since the selection is unordered, we can always write the 4 numbers in a nondecreasing order, say $a \leq b \leq c \leq d$. So we just need to find the cardinality of the following set*

$$\{(a, b, c, d) \in \mathbb{N}^4 \mid 1 \leq a \leq b \leq c \leq d \leq 13\}.$$

Note that the selection is very similar to a 4-combination, the only difference is that the numbers could be the same. In order to make sure that the numbers are distinct, we apply an operation as follows:

$$a' = a, b' = b + 1, c' = c + 2, d' = d + 3.$$

Then we have $1 \leq a' < b' < c' < d' \leq 13 + 3 = 16$. In fact, this operation admits a one-to-one correspondence between the above set and the following set

$$\{(a', b', c', d') \in \mathbb{N}^4 \mid 1 \leq a' < b < c < d \leq 16\}.$$

The new set exactly contains all 4-combinations on a 16-element set, so its cardinality is $\boxed{\binom{16}{4}} = 1,820$, *which is the answer.*

Extra Problem II (5 pts)

Prove that the set

$$\mathbb{Z}[\sqrt{2}] = \{a + b\sqrt{2} \mid a, b \in \mathbb{Z}\}$$

is a commutative ring with respect to the usual addition and multiplication.

Proof. First, we show that addition and multiplication are closed in $\mathbb{Z}[\sqrt{2}]$. For any integers a, b, c, d, we have

$$(a + b\sqrt{2}) + (c + d\sqrt{2}) = (a + c) + (b + d)\sqrt{2}$$

and

$$(a + b\sqrt{2}) \cdot (c + d\sqrt{2}) = (ac + 2bd) + (ad + bc)\sqrt{2}.$$

Since $a + c, b + d, ac + 2bd, ad + bc$ are also integers, the closedness is proved.

The commutativity, associativity and distributive laws remain for the usual addition and multiplication. Note that for any $a, b \in \mathbb{Z}$, we have

$$(a + b\sqrt{2}) + (0 + 0\sqrt{2}) = (0 + 0\sqrt{2}) + (a + b\sqrt{2}) = (a + b\sqrt{2}).$$

So $0 = 0 + 0\sqrt{2}$ is the identity for addition. And $-a, -b$ are also integers, so $-a + (-b)\sqrt{2} \in \mathbb{Z}[\sqrt{2}]$. And

$$(a + b\sqrt{2}) + (-a + (-b)\sqrt{2}) = 0.$$

So all elements have additive inverses.

Finally $1 = 1 + 0\sqrt{2} \in \mathbb{Z}[\sqrt{2}]$ and we have that for all $a, b \in \mathbb{Z}$,

$$(a + b\sqrt{2}) \cdot 1 = 1 \cdot (a + b\sqrt{2}) = a + b\sqrt{2}.$$

So 1 is the identity for multiplication. In summary, $\mathbb{Z}[\sqrt{2}]$ is indeed a commutative ring. □