Math 325K Fall 2018
Final Exam

December 13th, 2018

Name: ________________________________

I promise that I will abide by the UT Austin Honor Code while taking this exam.

Signature: ________________________________

Instructions:

- Time: 180 minutes.
- Score: 80+10 points. This exam counts 40% in your final grades.
- No textbooks, notes, cheat sheets, electronic devices allowed in this exam.
- You need to justify your answers for problems other than True/False and Multiple Choices.
- Please write your answers within the boxes on each page.
- You may request for more scratch papers.

1. (16 pts) True/False: each of the following arguments is either true or false and please mark your choice. You get 2 pts for each correct choice, 1 pt for **NOT** answering each question, and 0 pt for each incorrect/multiple choice. You **do not** need to justify your answer.

(1) Any conditional statement and its contraposition are logically equivalent.

True False

(2) To justify a universal statement, one eligible example is always enough.

True False

(3) For every positive integer n, if n^2 is even, then n must be even too.

True False

(4) Let A, B be subsets of the universal set U. Then $(A \cup B)^c = A^c \cup B^c$.

True False

(5) A sequence cannot be both arithmetic and geometric.

True False

(6) Let R be a relation defined on a set A. If R is symmetric and transitive, then it must be reflexive too.

True False

(7) If infinitely many objects are put into finitely many sets, there must be a set with infinitely many objects.

True False

(8) If one rolls a normal 6-side die once, the probability of the event "the outcome is a prime number" is less than the probability of the event "the outcome is an even number".

True False

2. (12 pts) Multiple choices: there is **exactly one** correct answer for each question. You get 4 pts for each correct choice, 1 pt for **NOT** answering each question, and 0 pt for each incorrect/multiple choice. You **do not** need to justify your answer.

(1) Here is a proof by mathematical induction of the statement "for all $n \in \mathbb{N}$, if $x, y \in \mathbb{N}$ such that $\max(x, y) = n$, then $x = y$".

Proof. We apply induction on n. Basis step is the case when $n = 1$. If $\max(x, y) = 1$, then both x and y are at most 1. Since they are positive integers, both must be 1 and they are equal. For the inductive step, suppose $k \in \mathbb{N}$ such that the statement is true for $n = k$, consider the case when $n = k + 1$. For $x, y \in \mathbb{N}$, if $\max(x, y) = k + 1$, then $\max(x - 1, y - 1) = k$. By the induction hypothesis, we have that $x - 1 = y - 1$. Hence $x = y$ and the inductive step is done. $\square$

What is the correct comment about the statement and the proof?

(a) The statement is correct and the proof is correct too.

(b) The statement is correct, but the proof is flawed because the basis step is incorrect.

(c) The statement is false, and the inductive step is false for $k = 1$ only.

(d) The statement is false, and the inductive step is false for all $k \geq 1$.

(2) Let $P(n)$ be an unary predicate. Suppose our task is to justify the statement "$\forall n \in \mathbb{N}, P(n)$", and we already justified $P(1)$ and "$\forall n \in \mathbb{N}, P(n) \to P(n+3)$". Which of the following statements is enough for our task if we can also justify it?

(a) $\forall n \in \mathbb{N},\ P(n+1) \to P(n)$.

(b) $\forall n \in \mathbb{N},\ P(n) \to P(2n)$.

(c) $\forall n \in \mathbb{N},\ P(n) \to P(n+2)$.

(d) $P(2)$.

(3) The logician Raymond Smullyan describes an island containing two types of people: knights who always tell the truth and knaves who always lie. Suppose you meet a group of three natives A, B, C on this island and they describe their identities to you as follows:

- A says: B is a knight.
- B says: I am the only knight among us.
- C says: There is at least one knave among us.

How many knights are there among them?

(a) 0.

(b) 1.

(c) 2.

(d) 3.

3. (4 pts)
Show that the statement form $(p \to q) \vee (q \to p)$ is a tautology.

4. (5 pts)

Let $A = \{1, 2, 3, 4, 5\}$ and the binary predicate $T(x, y)$ be "$x+y$ is a multiple of 3". Rephrase the following statement

$$\forall x \in A\, \forall y \in A\, \forall z \in A, (T(x, y) \wedge T(y, z)) \rightarrow T(x, z)$$

in an English sentence. Is it true or false? Justify your answer.

5. (4 pts)
Show that for every positive integer n, we have

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}.$$

6. (4 pts)
Show that for every positive integer n, we have $3^n > n^2 + 1$.

7. (5 pts)

Let $\{F_n\}_{n\geq 0}$ be the Fibonacci sequence. Show that for any integer $n \geq 0$, the terms F_n and F_{n+1} do not have a common divisor greater than 1 (in other words, for any integer $d > 1$, F_n and F_{n+1} cannot both be multiples of d).

8. (5 pts)
Let A, B, C, D be sets.

(1) (2 pts) Show that if $A \subseteq B$ and $C \subseteq D$, then $(A \cap C) \subseteq (B \cap D)$.

(2) (3 pts) The symbol $X \subsetneq Y$ means that X is a proper subset of Y. If $A \subsetneq B$ and $C \subsetneq D$, is it always true that $(A \cap C) \subsetneq (B \cap D)$? If yes, justify it; if no, present a counterexample.

9. (5 pts)

Suppose $f : \mathbb{N} \to \mathbb{N}$ is a function such that $f(f(x)) = x$ for all $x \in \mathbb{N}$. Show that f must be a one-to-one correspondence.

10. (4 pts)

A major airline in the U.S. has a list of its scheduled direct flights, for example AUS to DFW. This list defines a relation R on the set A of all airports that appear on this list as follows: for any two airports x and y in A, $(x, y) \in R$ if and only if there is a direct flight from x to y in the list. Based on your common sense, could R be an equivalence relation? Justify your answer.

11. (6 pts)

All of us have our own UT EIDs. The current pattern of a UT EID is either 3 lowercase letters followed by an integer from 0 to $9,999$, or 2 lowercase letters followed by an integer from 0 to $99,999$. Find the formula of the total number of eligible UT EIDs of current pattern. Justify your answer.

12. (6 pts)

On a weekday afternoon you wait for line 640 or 642 at a bus stop. According to the timetable, there are line 640 buses leaving at 1:05, 1:14 and 1:23 (every 9 minutes); and there are line 642 buses leaving at 1:06, 1:18 and 1:30 (every 12 minutes). You arrive x minutes past 1 o'clock, where x is a random integer from 1 to 30 with equal likelihood. You will take the next bus of either line that you can catch. For example, if you arrive at 1:10, you will catch the 640 bus at 1:14; if you arrive at 1:06, you can still catch the 642 bus. What is the probability that you end up catching a line 640 bus? Justify your answer.

13. (4 pts)

How many cards at least must one pick from a standard 52-card deck to be sure of getting at least 2 cards in the same suit? Justify your answer.

Extra Problem I (5 pts)

The "24 game" is an arithmetic card game. In each round, one deals a hand of four cards from a standard 52-card deck and takes the four corresponding numbers (for example $\spadesuit Q$ corresponds to 12, and $\diamondsuit 6$ corresponds to 6) then does arithmetic operations to get 24. The total number of possible hands is the total number of unordered selections of 4 integers from 1 to 13 where equal numbers are allowed (for example $1, 1, 7, 11$). Find this number and justify your answer. (Hint: the solution is not related to the number 24)

Extra Problem II (5 pts)

Prove that the set

$$\mathbb{Z}[\sqrt{2}] = \{a + b\sqrt{2} \mid a, b \in \mathbb{Z}\}$$

is a commutative ring with respect to the usual addition and multiplication.