Math 325K Fall 2018
Midterm Exam #1

October 4th, 2018

Name: _____

I promise that I will abide by the UT Austin Honor Code while taking this exam.

Signature: _____

Instructions:

- Time: 75 minutes.
- Score: 40+6 points. This exam counts 20% in your final grades.
- No textbooks, notes, cheat sheets, electronic devices allowed in this exam.
- You need to justify your answers for problems other than True/False and Multiple Choices.
- Please write your answers within the boxes on each page.
- You may request for more scratch papers.

The following table is a summary of some valid argument forms (rules of inference):

Name	Valid Argument Form
Modus Ponens	p $p \to q$ $\therefore q$
Modus Tollens	$p \to q$ $\sim q$ $\therefore \sim p$
Generalization	p $\therefore p \vee q$
Specialization	$p \wedge q$ $\therefore p$
Conjunction	p q $\therefore p \wedge q$
Elimination	$p \vee q$ $\sim q$ $\therefore p$
Transitivity	$p \to q$ $q \to r$ $\therefore p \to r$
Division into Cases	$p \vee q$ $p \to r$ $q \to r$ $\therefore r$
Contradiction Rule	$\sim p \to c(contradiction)$ $\therefore p$
Universal Modus Ponens	$\forall x \in D, P(x) \to Q(x)$ $a \in D$ $P(a)$ $\therefore Q(a)$
Universal Modus Tollens	$\forall x \in D, P(x) \to Q(x)$ $a \in D$ $\sim Q(a)$ $\therefore \sim P(a)$

1. (8 pts) True/False: each of the following arguments is either true or false and please mark your choice. You get 2 pts for each correct choice, 1 pt for **NOT** answering each question, and 0 pt for each incorrect/multiple choice. You **do not** need to justify your answer.

(1) The elements in a set are unordered.

True False

(2) The relation $C = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x = y^2\}$ is a function from $\mathbb{R}$ to $\mathbb{R}$.

True False

(3) The converse and the inverse of the same statement p are always logically equivalent.

True False

(4) When testing whether an argument form is valid, if the corresponding truth table has no critical row, then the argument form is invalid.

True False

2. (8 pts) Multiple choices: there is **exactly one** correct answer for each question. You get 4 pts for each correct choice, 1 pt for **NOT** answering each question, and 0 pt for each incorrect/multiple choice. You **do not** need to justify your answer.

(1) Which of the following statements is **NOT** logically equivalent to the negation of the statement "some spicy tacos are vegetarian"?

(a) All vegetarian tacos are not spicy.

(b) All non-spicy tacos are vegetarian.

(c) For any taco t, if t is spicy, then t is not vegetarian.

(d) For any taco t, t is not spicy or not vegetarian.

(2) Which of the following arguments is invalid?

(a) If I drank coffee last night, I could not fall asleep; I did fall asleep last night; $\therefore$ I didn't drink coffee last night.

(b) If Jane listens to music on the bus, then she is happy; if Jane reads a book on the bus, then she is happy; Jane either listens to music or reads a book on the bus; $\therefore$ Jane is happy.

(c) If John is at least 16 years old, then he can have a driver's license; If one has a driver's license, then he/she has a photo ID; John is 15 years old; $\therefore$ John does not have a photo ID.

(d) Any Canadian citizen can travel in the Schengen area without a visa for up to 90 days; Italy is a country in the Schengen area; Alice is a Canadian citizen; $\therefore$ Alice can travel in Italy without a visa for up to 90 days.

3. (4 pts) Prove that the statement form

$$p \to (q \to p)$$

is a tautology.

4. (5 pts) The Exclusive OR (XOR for short) is another binary logical connective, denoted by the symbol $\oplus$, its truth table is as follows:

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Question: Is the statement form $(p \oplus q) \oplus r$ logically equivalent to the statement form $p \oplus (q \oplus r)$? Justify your answer.

5. (4 pts) Prove that for any integers a and b, if 2 divides $a - b$, then 2 divides $a^2 + b^2$.

6. (6 pts) Let $D(x, y)$ be the binary predicate "x divides y". Rewrite the statement

$$\exists x \in \mathbb{N} \text{ such that } (x > 1) \wedge \left(\forall y \in \mathbb{N}, D(x, y^2) \rightarrow D(x, y)\right).$$

in an English sentence, and determine whether it is true or false. Justify your answer.

7. (5 pts) Let $n > 1$ be a composite number. Prove that there exists at least one prime divisor of n that is less than or equals to $\sqrt{n}$.

8. Extra Problem. (6 pts)

(1) (2 pt) Prove that for any two integers m and n, if $m \bmod 4 = n \bmod 4 = 1$, then $mn \bmod 4 = 1$.

(2) (4 pt) Prove that there exist infinitely many prime numbers p such that $p \bmod 4 = 3$. (Hint: proof by contradiction)