Math 325K Fall 2018
Midterm Exam #2 Solutions

Bo Lin

1. (8 pts) True/False: each of the following arguments is either true or false and please mark your choice. You get 2 pts for each correct choice, 1 pt for **NOT** answering each question, and 0 pt for each incorrect/multiple choice. You **do not** need to justify your answer.

(1) The common difference d of an arithmetic sequence could be any real number.

Solution. $\boxed{True}$. *$a_n = a_1 + (n-1)d$ is well-defined for any common difference $d \in \mathbb{R}$.*

(2) Let $n > r$ be positive integers. Then $\binom{n}{r} = \binom{n}{n-r}$.

Solution. $\boxed{True}$. *By definition,*

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

And thus

$$\binom{n}{n-r} = \frac{n!}{(n-r)!(n-(n-r))!} = \frac{n!}{(n-r)!r!}.$$

(3) There exist sets A and B such that $A - B$ and $B - A$ are NOT disjoint.

Solution. $\boxed{False}$. *For any element $x \in A - B$, by definition of difference set, $x \notin B$. Hence $x \notin B - A$. Then $A - B$ and $B - A$ are always disjoint.*

(4) If two functions are equal, then they must have the same co-domain.

Solution. $\boxed{False}$. *Two functions are equal if and only if they have the same domain and same value at every element of the domain. A particular counterexample could be the following:*

$$f : \mathbb{N} \to \mathbb{N}, f(x) = x \quad \forall x \in \mathbb{N}; g : \mathbb{N} \to \mathbb{Z}, g(x) = x \quad \forall x \in \mathbb{N}.$$

2. (8 pts) Multiple choices: there is **exactly one** correct answer for each question. You get 4 pts for each correct choice, 1 pt for **NOT** answering each question, and 0 pt for each incorrect/multiple choice. You **do not** need to justify your answer.

(1) Here is an example of **incorrect** proof by mathematical induction. Statement: for every positive integer n, every n people have the same name.

Proof. Basis step: when $n = 1$, the conclusion is trivially true. Inductive step: suppose $m \geq 1$ is an arbitrary integer such that the argument is true for $n = m$. Consider the case when $n = m + 1$. For any $m + 1$ people $p_1, \ldots, p_{m+1}$, by the induction hypothesis, the m people $p_1, \ldots, p_m$ have the same name, and the m people $p_2, \ldots, p_{m+1}$ also have the same name, so all $m + 1$ people $p_1, \ldots, p_{m+1}$ have the same name, the inductive step is done. □

What is the flaw in this 'proof' of the absurd statement?

(a) the basis step is wrong;

(b) the inductive step is wrong for all $m \geq 1$;

(c) the inductive step is wrong for $m = 1$ only;

(d) the inductive hypothesis is wrongly stated.

Solution. *First, the statement is wrong, so there must be a flaw in the proof. The basis step is correct, as for every group of one person, the people in the group have the same name. For the inductive step, the inductive hypothesis is stated correctly, that every m people have the same. Note that in fact it is not true that 'every two people have the same name', so the inductive step is definitely wrong when $m = 1$. While it is true when $m \geq 2$, as the hypothesis for $m = 2$ is 'every two people have the same name', which already implies that 'all people have the same name'. So the answer is $\boxed{(c)}$.*

(2) In comparison to the ordinary principle of mathematical induction, why is the alternative principle called "strong" mathematical induction? (for convenience let n be the statement variable)

(a) Because more cases of n are justified in the basis step.

(b) Because more cases of n are included in the inductive hypothesis.

(c) Because more cases of n are justified in the inductive step.

(d) Because more cases of n are included in the conclusion that we need to justify.

Solution. *The answer is* $\boxed{(b)}$. *In the ordinary version, the inductive hypothesis only contains one previous case of the universal statement, say* $P(m)$*; while in the strong version, the inductive hypothesis contains* $P(i)$ *for all* $a \leq i \leq m$*, where* a *is the very first case covered by the basis step. For (a), even in the strong version, it is possible that there is still only one case in the basis step; for (c), we always justify one case in the inductive step; for (d), the conclusion we need to justify would be the same too.*

3. (4 pts)
Show that for every positive integer n, we have the following identity:

$$\sum_{i=1}^{n} i(i+1) = \frac{n(n+1)(n+2)}{3}.$$

Proof. We apply mathematical induction on n. The basis step is when $n = 1$. The left hand side becomes $1 \cdot 2 = 2$, and the right hand side becomes $\frac{1 \cdot 2 \cdot 3}{3} = 2$, so the basis step is done. For the inductive step, suppose m is a positive integer such that the statement is true for $n = m$, then we have the inductive hypothesis that

$$\sum_{i=1}^{m} i(i+1) = \frac{m(m+1)(m+2)}{3}.$$

Now we consider the case when $n = m + 1$. We have

$$\begin{aligned}
\sum_{i=1}^{m+1} i(i+1) &= \sum_{i=1}^{m} i(i+1) + (m+1)(m+2) \\
&= \frac{m(m+1)(m+2)}{3} + (m+1)(m+2) \\
&= (m+1)(m+2)\left(\frac{m}{3} + 1\right) \\
&= (m+1)(m+2)\frac{m+3}{3} \\
&= \frac{(m+1)(m+2)(m+3)}{3}.
\end{aligned}$$

Hence the statement is also true for $n = m+1$, the inductive step is done. □

4. (4 pts)
Let $\{F_n\}_{n\geq 0}$ be the Fibonacci sequence. Show that for every positive integer n,

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n.$$

Proof. We apply mathematical induction on n. The basis step is when $n = 1$. Since $F_0 = 0, F_1 = F_2 = 1$, we have

$$F_2 F_0 - F_1^2 = 0 - 1 = -1 = (-1)^1.$$

The basis step is done.

As for the inductive step, suppose m is a positive integer such that the statement is true for $n = m$, then we have the inductive hypothesis that

$$F_{m+1} F_{m-1} - F_m^2 = (-1)^m.$$

Now we consider the case when $n = m + 1$. By the recursive relation of the Fibonacci sequence, we have

$$F_{m+2} = F_{m+1} + F_m$$

and

$$F_{m+1} = F_m + F_{m-1}.$$

So

$$\begin{aligned} F_{m+2} F_m - F_{m+1}^2 &= (F_{m+1} + F_m) F_m - F_{m+1}^2 \\ &= F_{m+1} F_m + F_m^2 - F_{m+1}^2 \\ &= F_m^2 - F_{m+1} (F_{m+1} - F_m) \\ &= F_m^2 - F_{m+1} F_{m-1} = -(-1)^m \\ &= (-1)^{m+1}. \end{aligned}$$

And the inductive step is done.

□

5. (4 pts)
The Lucas sequence $\{L_n\}_{n \geq 0}$ is defined as follows:

$$L_1 = 1, L_2 = 3, L_{n+2} = L_{n+1} + L_n \quad \forall n \in \mathbb{N}.$$

Show that $L_n < \left(\frac{7}{4}\right)^n$ for all positive integers n.

Proof. We apply the strong mathematical induction on n. For the basis step we consider the cases of $n = 1$ and $n = 2$. We have

$$L_1 = 1 < \frac{7}{4} = \left(\frac{7}{4}\right)^1$$

and

$$L_2 = 3 < \frac{49}{16} = \left(\frac{7}{4}\right)^2.$$

The basis step is done.

As for the inductive step, suppose $m \geq 2$ is a positive integer such that the statement is true for all integers i with $1 \leq i \leq m$, then $L_i < \left(\frac{7}{4}\right)^i$ for all $1 \leq i \leq m$. Now we consider the case when $n = m+1$. Since $m \geq 2$, by definition,

$$L_{m+1} = L_m + L_{m-1}.$$

Note that $1 \leq i \leq m$ for both $i = m-1$ and $i = m$, then

$$L_m < \left(\frac{7}{4}\right)^m, L_{m-1} < \left(\frac{7}{4}\right)^{m-1}.$$

In addition, we have

$$\frac{7}{4} + 1 = \frac{11}{4} = \frac{44}{16} < \frac{49}{16} = \left(\frac{7}{4}\right)^2.$$

Hence

$$L_{m+1} < \left(\frac{7}{4}\right)^m + \left(\frac{7}{4}\right)^{m-1} = \left(\frac{7}{4}\right)^{m-1}\left(\frac{7}{4} + 1\right) < \left(\frac{7}{4}\right)^{m-1} \cdot \left(\frac{7}{4}\right)^2 = \left(\frac{7}{4}\right)^{m+1}.$$

And the inductive step is done. □

6. (5 pts)
Let A, B, C, D be sets.

(1) (2 pts) Show that if $A \subseteq B$ and $C \subseteq D$, then $(A \cup C) \subseteq (B \cup D)$.

(2) (3 pts) The symbol $X \subsetneq Y$ means that X is a proper subset of Y. If $A \subsetneq B$ and $C \subsetneq D$, is it always true that $(A \cup C) \subsetneq (B \cup D)$? If yes, justify it; if no, present a counterexample.

Solution. *(1) It suffices to show that for any $x \in A \cup C$, we also have $x \in B \cup D$. By definition of union, $x \in A$ or $x \in C$. We divide into cases. If $x \in A$, since $A \subseteq B$, we have $x \in B$, thus $x \in B \cup D$; if $x \in C$, since $C \subseteq D$, we have $x \in D$, thus $x \in B \cup D$. So in either case we have $x \in B \cup D$.*

(2) The answer is $\boxed{no}$. One counterexample would be the following:

$$A = \{1\}, C = \{2\}, B = D = \{1, 2\}.$$

Then A is a proper subset of B and C is a proper subset of D, while

$$A \cup C = B \cup D = \{1, 2\},$$

so $A \cup C$ is not a proper subset of $B \cup D$.

7. (7 pts)

Let Z_4 be the set $\{0, 1, 2, 3, 4\}$. We define a function $Fr : Z_4 \to Z_4$ such that for all $x \in Z_4$,

$$Fr(x) = x^5 \bmod 5.$$

(1) (2 pts) Explain why Fr is well-defined.
(2) (2 pts) Find $Fr^{-1}(2)$, the inverse image of 2.
(3) (3 pts) Is Fr a one-to-one function? Justify your answer.

Solution. *(1) Because for each element x in the domain Z_4 of Fr, $Fr(x) = x^5 \bmod 5$ is indeed an integer between 0 and 4, which belongs to the co-domain of Fr.*

(2) We have the following table:

x	0	1	2	3	4
x^5	0	1	32	243	1024
$x^5 \bmod 5$	0	1	2	3	4

So the only x in Z_4 such that $Fr(x) = 2$ is $x = 2$, and $Fr^{-1}(2) = \boxed{\{2\}}$.

(3) According to the table above, $Fr(x)$ has distinct values for difference x in the domain of Fr, by definition it $\boxed{\text{is}}$ one-to-one.

Remark 1. *The map is from the famous Frobenius endomorphism that has been widely used in albegraic number theory and other related fields of mathematics.*

8. Extra Problem. (6 pts)

For any two sets A and B, their *symmetric difference* is defined as follows:

$$A \triangle B = (A - B) \cup (B - A)\,.$$

(1) (2 pts) Show that for any two sets A and B, we have

$$A - (A \triangle B) = B - (A \triangle B)\,.$$

(2) (4 pts) Let A, B and C be sets. Suppose $A \triangle C = B \triangle C$. Is it always true that $A = B$? If yes, justify it; if no, present a counterexample.

(Hint: Venn diagrams may be helpful.)

Solution. *(1) It suffices to show that*

$$A - (A \triangle B) = A \cap B,$$

because then by symmetry we also have

$$B - (A \triangle B) = A \cap B.$$

For any $x \in A - (A \triangle B)$, by definition $x \in A$ and $x \notin (A \triangle B)$. Then $x \notin A - B$. Suppose $x \notin B$, by definition $x \in A - B$, a contradiction! Hence $x \in B$, so $x \in A \cap B$. We justified that

$$A - (A \triangle B) \subseteq A \cap B.$$

Conversely, for any $y \in A \cap B$, we have $y \in A$ and $y \in B$. By definition of difference set, $y \notin A - B$ and $y \notin B - A$, then $y \notin A \triangle B$. By definition of difference, we have $y \in A - (A \triangle B)$, hence

$$A \cap B \subseteq A - (A \triangle B).$$

We are done.

(2) The answer is $\boxed{yes}$. By the symmetry of A and B, it suffices to show that $A \subseteq B$, which is equivalent to "for any $x \in A$, $x \in B$". Suppose $x \in A$. Note that either $x \in C$ or $x \notin C$, we divide into cases.

If $x \in C$, then by the same argument as in (1), $x \notin A \triangle C$. Hence $x \notin B \triangle C$. In particular, $x \notin C - B$. While $x \in C$ already holds, then $x \notin B$ cannot hold and thus $x \in B$.

If $x \notin C$, then $x \in A - C \subseteq A \triangle C$ and thus $x \in A \triangle C$. Hence $x \in B \triangle C$. BY definition of symmetric difference, $x \in B - C$ or $x \in C - B$. Since $x \notin C$, we have $x \notin C - B$. So it must be the case that $x \in B - C$, and $x \in B$.

So in either case we have $x \in B$, we are done.