Math 325K Fall 2018
Midterm Exam #2

November 6th, 2018

Name: ________________________________

I promise that I will abide by the UT Austin Honor Code while taking this exam.

Signature: ________________________________

Instructions:

- Time: 75 minutes.
- Score: 40+6 points. This exam counts 20% in your final grades.
- No textbooks, notes, cheat sheets, electronic devices allowed in this exam.
- You need to justify your answers for problems other than True/False and Multiple Choices.
- Please write your answers within the boxes on each page.
- You may request for more scratch papers.

For all subsets A, B, C of a universal set U, we have the following identities.

- (Commutative Laws) $A \cup B = B \cup A$; $A \cap B = B \cap A$.
- (Associative Laws) $A \cup (B \cup C) = (A \cup B) \cup C$; $A \cap (B \cap C) = (A \cap B) \cap C$.
- (Distributive Laws) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$; $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
- (Identity Laws) $A \cup \emptyset = A \cap U = A$.
- (Complement Laws) $A \cup A^c = U$; $A \cap A^c = \emptyset$.
- (Double Complement Law) $(A^c)^c = A$.
- (Idempotent Laws) $A \cup A = A \cap A = A$.
- (Universal Bound Laws) $A \cup U = U$; $A \cap \emptyset = \emptyset$.
- (De Morgans Laws) $(A \cup B)^c = A^c \cap B^c$; $(A \cap B)^c = A^c \cup B^c$.
- (Absorption Laws) $A \cup (A \cap B) = A \cap (A \cup B) = A$.
- (Complements of U and $\emptyset$) $U^c = \emptyset$; $\emptyset^c = U$.
- (Set Difference Law) $A - B = A \cap B^c$.

1. (8 pts) True/False: each of the following arguments is either true or false and please mark your choice. You get 2 pts for each correct choice, 1 pt for **NOT** answering each question, and 0 pt for each incorrect/multiple choice. You **do not** need to justify your answer.

(1) The common difference d of an arithmetic sequence could be any real number.

True False

(2) Let $n > r$ be positive integers. Then $\binom{n}{r} = \binom{n}{n-r}$.

True False

(3) There exist sets A and B such that $A - B$ and $B - A$ are NOT disjoint.

True False

(4) If two functions are equal, then they must have the same co-domain.

True False

2. (8 pts) Multiple choices: there is **exactly one** correct answer for each question. You get 4 pts for each correct choice, 1 pt for **NOT** answering each question, and 0 pt for each incorrect/multiple choice. You **do not** need to justify your answer.

(1) Here is an example of **incorrect** proof by mathematical induction. Statement: for every positive integer n, every n people have the same name.

Proof. Basis step: when $n = 1$, the conclusion is trivially true. Inductive step: suppose $m \geq 1$ is an arbitrary integer such that the argument is true for $n = m$. Consider the case when $n = m + 1$. For any $m + 1$ people $p_1, \ldots, p_{m+1}$, by the induction hypothesis, the m people $p_1, \ldots, p_m$ have the same name, and the m people $p_2, \ldots, p_{m+1}$ also have the same name, so all $m + 1$ people $p_1, \ldots, p_{m+1}$ have the same name, the inductive step is done. □

What is the flaw in this 'proof' of the absurd statement?

(a) the basis step is wrong;

(b) the inductive step is wrong for all $m \geq 1$;

(c) the inductive step is wrong for $m = 1$ only;

(d) the inductive hypothesis is wrongly stated.

(2) In comparison to the ordinary principle of mathematical induction, why is the alternative principle called "strong" mathematical induction? (for convenience let n be the statement variable)

(a) Because more cases of n are justified in the basis step.

(b) Because more cases of n are included in the inductive hypothesis.

(c) Because more cases of n are justified in the inductive step.

(d) Because more cases of n are included in the conclusion that we need to justify.

3. (4 pts)
Show that for every positive integer n, we have the following identity:

$$\sum_{i=1}^{n} i(i+1) = \frac{n(n+1)(n+2)}{3}.$$

4. (4 pts)

Let $\{F_n\}_{n\geq 0}$ be the Fibonacci sequence. Show that for every positive integer n,

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n.$$

5. (4 pts)

The Lucas sequence $\{L_n\}_{n\geq 0}$ is defined as follows:

$$L_1 = 1, L_2 = 3, L_{n+2} = L_{n+1} + L_n \quad \forall n \in \mathbb{N}.$$

Show that $L_n < \left(\frac{7}{4}\right)^n$ for all positive integers n.

6. (5 pts)
Let A, B, C, D be sets.

(1) (2 pts) Show that if $A \subseteq B$ and $C \subseteq D$, then $(A \cup C) \subseteq (B \cup D)$.

(2) (3 pts) The symbol $X \subsetneq Y$ means that X is a proper subset of Y. If $A \subsetneq B$ and $C \subsetneq D$, is it always true that $(A \cup C) \subsetneq (B \cup D)$? If yes, justify it; if no, present a counterexample.

7. (7 pts)

Let Z_4 be the set $\{0, 1, 2, 3, 4\}$. We define a function $Fr : Z_4 \to Z_4$ such that for all $x \in Z_4$,

$$Fr(x) = x^5 \bmod 5.$$

(1) (2 pts) Explain why Fr is well-defined.

(2) (2 pts) Find $Fr^{-1}(2)$, the inverse image of 2.

(3) (3 pts) Is Fr a one-to-one function? Justify your answer.

8. Extra Problem. (6 pts)

For any two sets A and B, their *symmetric difference* is defined as follows:

$$A \triangle B = (A - B) \cup (B - A) .$$

(1) (2 pts) Show that for any two sets A and B, we have

$$A - (A \triangle B) = B - (A \triangle B) .$$

(2) (4 pts) Let A, B and C be sets. Suppose $A \triangle C = B \triangle C$. Is it always true that $A = B$? If yes, justify it; if no, present a counterexample.

(Hint: Venn diagrams may be helpful.)