Math 325K - Lecture 1
Chapter 1

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Outline

- Variables and statements.
- Sets.
- Relations and functions.

Variables

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- to represent an arbitrary element in a given set where we do not want to fix a particular element.

Remark

Since variables are placeholders, we can use whatever symbols we want as long as they are consistent.

Example of variable

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Solution

Is there a number x such that $2x + 3 = x^2$?

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Solution

For all real numbers r, $r^2 \geq 0$.

Various statements

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Definition

Given a property, an **existential statement** *says that there is at least one thing for which the property is true.*

Examples of statements

Let's figure out the type of following statements

Example

1. *All positive numbers are greater than zero.*
2. *There is a prime number that is even.*
3. *If an integer n is not divisible by 3, then $n^2 - 1$ is divisible by 3.*

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Solution

1. *Universal statement.*
2. *Existential statement.*
3. *Conditional statement.*

Universal conditional statements

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Example

For all animals a, if a is a dog, then a is a mammal.

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Example

For all positive integer x, there exists a positive integer y such that y is greater than x.

Existential universal statements

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Example

There exists a positive integer x such that for all positive integer y, y is divisible by x.

Remark

This statement is true because $x = 1$ works.

What is a set

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Axiom

The **axiom of extension** *says that a set is completely determined by what its elements are - not the order in which they might be listed or the fact that some elements might be listed more than once.*

Set-roster notation

If we can list all elements $x_1, x_2, \ldots$ of a set S, we can use the following notation:

$$S = \{x_1, x_2, \ldots\}.$$

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Remark

Set-roster notation works no matter whether a set contains finitely many or infinitely many elements, while in the latter case we should make the pattern of elements very clear as it is impossible to present all of them.

Set-builder notation

If a set contains too many elements, or it is defined implicitly by some properties, we may use another notation. Let S be a set and let $P(x)$ be a property that elements of S may or may not satisfy. We may define a new set to be the set of all elements x in S such that $P(x)$ is true. We denote this set as follows:

$$\{x \in S \mid P(x)\}.$$

This notation is called the **set-builder notation**.

Remark

In set-builder notation, within the pair of curly brackets, the space is divided into two parts by a vertical bar $|$*. On the left is the description of a general element of the set, and on the right is the properties all elements must satisfy.*

Notations of some important sets

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Symbol	Meaning
$\mathbb{Z}$	the set of all integers
$\mathbb{N}$	the set of all positive integers
$\mathbb{Q}$	the set of all rational numbers
$\mathbb{R}$	the set of all real numbers
$\mathbb{C}$	the set of all complex numbers

Example of notations of sets

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Denote the set of all even integers not exceeding 10 *using both set-roster notation and set-builder notation.*

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Solution

Set-roster notation: $\{2, 4, 6, 8, 10\}$.
Set-builder notation:

$$\{x \in \mathbb{N} \mid x \text{ is even } \& \quad x \leq 10\}$$

or

$$\{2k \mid k \in \mathbb{Z}, 1 \leq k \leq 5\}.$$

Subsets

Definition

If A *and* B *are sets, then* A *is called a* **subset** *of* B*, written* $A \subseteq B$*, if and only if every element of* A *is also an element of* B*; otherwise we write* $A \not\subseteq B$.

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If A and B are sets, then A is called a **proper subset** *of B, written $A \subsetneq B$, if and only if every element of A is in B but there is at least one element of B that is not in A.*

Example of subsets

Example

Among the sets

$$\mathbb{N}, \mathbb{Q}, \mathbb{R}, \mathbb{Z},$$

find as many as possible pairs of A *and* B *such that* $A \subseteq B$.

Example of subsets

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find as many as possible pairs of A and B such that $A \subseteq B$.

Solution

We have

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}.$$

Cartesian products

Definition

Given elements a and b, the symbol (a, b) denotes the ordered pair consisting of a and b together with the specification that a is the first element of the pair and b is the second element. Two ordered pairs (a, b) and (c, d) are equal if and only if $a = c$ and $b = d$.

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Definition

Given sets A and B, the Cartesian product of A and B, denoted $A \times B$ and read A cross B, is the set of all ordered pairs (a, b), where a is in A and b is in B:

$$A \times B = \{(a, b) \mid a \in A, b \in B\}.$$

Example of a Cartesian product

Example

Let $A = \{2, 4, 6\}$ *and* $B = \{1, 3, 5\}$. *Denote their Cartesian product* $A \times B$ *and find the number of elements in it.*

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Solution

$$A \times B = \{(2,1),(2,3),(2,5),(4,1),(4,3),(4,5),(6,1),(6,3),(6,5)\}.$$

So there are $3 \cdot 3 = 9$ *elements in their Cartesian product.*

Cardinality

Definition

The **cardinality** *of a set A, written $|A|$, is the number of elements in the set. If A contains finitely many elements, then its cardinality is a nonnegative integer; otherwise $|A|$ is an infinite quantity (Warning: those infinite quantities have very interesting yet complicated properties and made several preeminent mathematicians in history lose their minds after pondering for a long time).*

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Proposition

Let A and B be finite sets. Then

$$|A \times B| = |A| \cdot |B|.$$

Relations

Definition

Let A and B be sets. A **relation** *R from A to B is a subset of $A \times B$. Given an ordered pair (x, y) in $A \times B$, x is related to y by R, written xRy, if and only if (x, y) is in R. The set A is called the* **domain** *of R and the set B is called its* **co-domain**.

Example of relations

Example

Let $A = \{1, 2\}$ *and* $B = \{1, 2, 3\}$ *be sets, and define a relation* R *from* A *to* B *as follows: given any* $(x, y) \in A \times B$, xRy *if and only if* $\frac{x-y}{2} \in \mathbb{Z}$. *Write* R *as a set and find its domain and co-domain.*

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Solution

$$A \times B = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3)\}.$$

By definition,

$$R = \{(1,1), (1,3), (2,2)\}.$$

And its domain is A, *its co-domain is* B.

Functions

Functions are a special kind of relation that each element in A is related to a unique element in B.

Definition

A **function** F *from a set* A *to a set* B *is a relation with domain* A *and co-domain* B *that satisfies the following two properties:*

1. *for every element* $x \in A$*, there is an element* $y \in B$ *such that* $(x, y) \in F$*;*
2. *if* $(x, y) \in F$ *and* $(x, z) \in F$*, then* $y = z$*.*

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Remark

For $x \in A$, *by definition there is a unique* $y \in B$ *such that* $(x, y) \in F$. *This* y *is usually denoted as* $F(x)$.

Example of functions

Example

Let $A = B = \mathbb{R}$ *and* F *be the relation*

$$\{(x, x^2) \mid x \in \mathbb{R}\}.$$

Is F *a function?*

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Solution

For each $x \in A = \mathbb{R}$, x *is related to a unique element* $x^2 \in \mathbb{R} = B$ *in* F, *so* F *is a function.*

Example of functions

Example

Let $A = B = \mathbb{R}$ *and* C *be the relation of the unit circle:*

$$C = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x^2 + y^2 = 1\}.$$

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What happens when $x = 1$*? when* $x = 0$*?*

Solution

When $x = 0 \in A$, x *is related to elements* $\pm 1 \in \mathbb{R} = B$ *in* C, *so* C *is not a function.*