Math 325K - Lecture 10
Section 4.6 & Review

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Outline

- Some square roots of integers are irrational numbers.
- The cardinality of the set of prime numbers.
- Review of Midterm # 1.

Reduced fractions

Proposition

For any rational number x, there exist integers m and n such that $x = \frac{m}{n}$ and m, n do not have a common divisor greater than 1.

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Sketch of proof.

Since x is rational, there exists integers a, b such that $x = \frac{a}{b}$. If a and b have a common divisor $d > 1$, then both a/d and b/d are integers, so $x = \frac{a/d}{b/d}$. If a/d and b/d still have a common divisor greater than 1, then we can repeat the procedure. By the "well-ordering principle for the integers" (we will discuss in Section 5.4), this procedure will terminate after finitely many steps and we are done. $\square$

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Definition

Such a fraction is called the **reduced form** *of a rational number.*

Example: reduced fractions

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For the following rational numbers, write them in the reduced form:

(a) 1.5;
(b) $-\frac{6}{15}$;
(c) $2 + \frac{3}{4}$.

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(a) $\frac{3}{2}$. *(b)* $\frac{-2}{5}$. *(c)* $\frac{11}{4}$.

Pythagorean Theorem

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Theorem (Pythagorean Theorem)

In a right-angled triangle, suppose the skew side has length c and the other two sides have lengths a and b respectively, then

$$c^2 = a^2 + b^2.$$

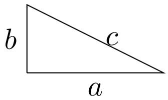

Figure: A right-angled triangle

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If we take $a = b = 1$ in the triangle, then Pythagorean Theorem tells us that $c^2 = 2$, in other words the square of c is 2.

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Remark

Apparently c cannot be any integer, what about rational numbers?

The very first irrational number discovered

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Theorem

$\sqrt{2}$ *is irrational.*

The proof

Here we introduce a proof by Aristotle:

Proof.

We prove by contradiction. Suppose $\sqrt{2}$ is rational, then it has a reduced form, which is $\frac{m}{n}$ where m, n are integers that do not have a common divisor greater than 1. We have $\frac{m^2}{n^2} = (\frac{m}{n})^2 = (\sqrt{2})^2 = 2$. So $m^2 = 2n^2$ is even. Then m is even. So there exists an integer k such that $m = 2k$. In addition, since m, n do not have a common divisor greater than 1 and $2|m$, n must not be divisible by 2, so n is odd. However, we have $2n^2 = m^2 = (2k)^2 = 4k^2$, so $n^2 = 2k^2$ is even, too. Then n is even, a contradiction! We are done. □

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The general case

In general we have the following theorem.

Theorem

Let n, k be positive integers such that n is not the k-th power of any integer, then $\sqrt[k]{n}$ is irrational.

Consecutive integers have no common prime divisor

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Let p be a prime number and a be an integer. If $p \mid a$, then $p \nmid (a+1)$.

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Proof.

We prove by contradiction. Suppose $p \mid (a+1)$, then there is an integer s such that $a+1 = ps$. Since $p \mid a$, there is an integer r such that $a = pr$. Then

$$1 = (a+1) - a = ps - pr = p(s-r).$$

So p is a divisor of 1, a contradiction! We are done. □

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Proof.

We prove by contradiction. Suppose there are only finitely many prime numbers, then we may list them as: (here $n \in \mathbb{N}$)

$$2 = p_1 < p_2 < p_3 < \cdots < p_n.$$

Now we let M be the product of all p_i's and we consider the divisors of the number $M + 1$. □

Proof continued

Proof.

(continued) For each $1 \leq i \leq n$, we have $p_i \mid M$. By the previous proposition, p_i does not divide $M+1$. However, since $M+1$ is greater then 1, it has at least one prime divisor, a contradiction! We are done. □

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Remark

There are many other methods to prove this theorem. In particular, Riemann consider the Zeta function when considering the distribution of prime numbers in $\mathbb{N}$, and led to the famous Riemann Conjecture.

Midterm # 1 Review Outline

Sets, relations and functions

- Sets - each element appears only once, elements are not ordered.
- Relations - subsets of Cartesian products.
- Functions - each element in the domain corresponds to exactly one element in the co-domain.

Logical statements

- The logical connectives $\sim, \wedge, \vee, \rightarrow, \leftrightarrow$.
- Truth values and truth tables.
- Logical equivalence, tautology and contradiction.
- Contrapositive, converse and inverse, common errors.

Arguments and validity

- Definition of argument and argument form.
- Definition of validity.
- Test validity using truth tables, critical row.
- Difference between "true" and "valid", definition of sound.
- Rules of inference. (provided on the exam paper)

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Predicates and quantifiers

- Definitions of them.
- Meanings of $\forall$ and $\exists$.
- Related forms (negation etc.) of quantified statements.
- Multi-quantified statements.
- Quantified rules of inference.

Methods of proof

- Direct proof.
- Proof by example or counterexample.
- Proof by division into cases.
- Proof by contradiction.
- Proof by contraposition.

Elementary number theory

- Even and odd numbers.
- Prime and composite numbers.
- Divisors and multiples.
- Rational and irrational numbers.