Math 325K - Lecture 11
Section 5.1 Sequences

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October 9th, 2018

Outline

- Sequences.
- Summation notation $\sum$ and product notation $\prod$.
- Factorial and "n choose r" notation.

Motivation

Sometimes we would like to study the pattern of repeated processes, and sequences are the main mathematical structure we use.

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Definition

A **sequence** *is a function whose domain is either all the integers between two given integers ($\{x \in \mathbb{Z} \mid a \leq x \leq b\}$) or all the integers greater than or equal to a given integer ($\{x \in \mathbb{Z} \mid x \geq a\}$).*

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Remark

In notation, we usually write a sequence as $\{a_k\}$, where for every integer k in the domain, a_k is the image of k. k is called an **index** *and a_k is called a* **term**.

Explicit formula

Since each sequence is a function, we would like to know what is the correspondence between its domain and co-domain. An **explicit formula** for a sequence is a rule that shows how the values of a_k depend on k.

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The sequence $1, 2, 3, 4, \cdots$ *has the explicit formula* $a_k = k$ *for* $k \in \mathbb{N}$.

Remark

In some cases, it is very hard or even impossible to present an explicit formula for a sequence. For example, let p_k *be the* k*-th smallest prime number.*

Example: find terms of sequences given by explicit formulas

Example

Let a_k be a sequence with domain $\mathbb{N}$ and explicit formula $a_k = \frac{1}{k(k+1)}$ for all $k \in \mathbb{N}$. Find

(a) a_1;

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Solution

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Solution

(a) $a_1 = \frac{1}{1 \cdot 2} = \frac{1}{2}$.
(b) $a_4 = \frac{1}{4 \cdot 5} = \frac{1}{20}$.

Example: find explicit formula given terms of a sequence

Example

Find an explicit formula for the following sequences:

- (a) *domain* $= \{1, 2, 3, 4\}$, $a_1 = -1, a_2 = 1, a_3 = -1, a_4 = 1$;
- (b) *domain* $= \mathbb{N}$, $a_1 = 1, a_2 = 3, a_3 = 5, \cdots$;
- (c) *domain* $= \mathbb{N}$, $a_1 = 3, a_2 = 9, a_3 = 27, a_4 = 81, \cdots$.

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Solution

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Solution

(a) $a_k = (-1)^k$ *for* $1 \leq k \leq 4, k \in \mathbb{Z}$.
(b) $a_k = 2k - 1$ *for* $k \in \mathbb{N}$.

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(a) $a_k = (-1)^k$ *for* $1 \leq k \leq 4, k \in \mathbb{Z}$.
(b) $a_k = 2k - 1$ *for* $k \in \mathbb{N}$.
(c) $a_k = 3^k$ *for* $k \in \mathbb{N}$.

Sum of terms in a sequence

Given a sequence, one natural thing to consider is the sum of its terms, especially when there are finitely many terms. Of course we can write the sum as

$$a_1 + a_2 + \cdots + a_n.$$

But this is inconvenient, and when there are infinitely many terms, it becomes impossible to cover all terms in this notation. So we need some better notation.

The big Sigma notation

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Definition

Let $m \leq n$ be integers. The big Sigma notation is

$$\sum_{k=m}^{n} a_k = a_m + a_{m+1} + \cdots + a_n.$$

Here k is the index of the summation, m is called the **lower limit** *of the summation and n is called the* **upper limit** *of the summation. The right hand side of above formula is called the* **expanded form** *of the sum.*

Dummy variable

Recall that in quantified statements, the symbol of the statement variables is not important. For example, "$\forall x, P(x)$" and "$\forall y, P(y)$" represent exactly the same statement. We have the same pattern in $\sum$.

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Remark

In $\sum_{k=m}^{n} a_k$, the index k is also called a **dummy variable**, *because one can replace it with any other letter. For example, $\sum_{i=m}^{n} a_i$ expresses the same summation.*

Example: computing summations

Example

Compute the following summations of terms in sequences:

(a)
$$\sum_{k=3}^{6} a_k, \quad a_k = (-1)^k \quad \forall k \in \mathbb{N}.$$

(b)
$$\sum_{k=1}^{5} k^2.$$

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Solution

(a) The sum is $a_3 + a_4 + a_5 + a_6 = (-1) + 1 + (-1) + 1 = 0.$

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Solution

(a) The sum is $a_3 + a_4 + a_5 + a_6 = (-1) + 1 + (-1) + 1 = 0.$
(b) The sum is $1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 1 + 4 + 9 + 16 + 25 = 55.$

General $\sum$ notations

The big Sigma notation can also be used to express more complicated sums, where the indices are not limited to consecutive integers.

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The following notation

$$\sum_{\substack{p\leq 10 \\ p \text{ is prime}}} p^2$$

is summing over all indices p that satisfy the descriptions, which are $2, 3, 5, 7$ in this example, so the sum is

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The big Pi notation

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Definition

Let $m \leq n$ be integers. The big Pi notation is

$$\prod_{k=m}^{n} a_k = a_m \cdot a_{m+1} \cdot \ldots \cdot a_n.$$

Here k is the index of the product, m is called the **lower limit** *of the product and n is called the* **upper limit** *of the product.*

Example: compute the products

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Compute the following products of terms in sequences:

$$(a) \quad \prod_{k=1}^{5} k; \quad (b) \quad \prod_{k=1}^{10} \frac{k}{k+1}.$$

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Solution

(a) The product is $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$.

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$$(a) \quad \prod_{k=1}^{5} k; \quad (b) \quad \prod_{k=1}^{10} \frac{k}{k+1}.$$

Solution

(a) The product is $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$.
(b) The product is a rational number, whose numerator is the product of integers from 1 *to* 10 *and the denominator is the product of integers from* 2 *to* 11*, so after a lot of cancellations we get* $\frac{1}{11}$.

Useful properties

Theorem

If $a_m, a_{m+1}, a_{m+2}, \cdots$ and $b_m, b_{m+1}, b_{m+2}, \cdots$ are sequences of real numbers and c is any real number, then the following equations hold for any integer $m \leq n$:

(a)
$$\sum_{k=m}^{n} a_k + \sum_{k=m}^{n} b_k = \sum_{k=m}^{n} (a_k + b_k).$$

(b)
$$c \cdot \sum_{k=m}^{n} a_k = \sum_{k=m}^{n} c \cdot a_k.$$

(c)
$$\left(\prod_{k=m}^{n} a_k\right) \cdot \left(\prod_{k=m}^{n} b_k\right) = \prod_{k=m}^{n} (a_k \cdot b_k).$$

Factorial

Definition

For $n \in \mathbb{N}$, the quantity of n **factorial**, *denoted by $n!$, is defined as the product*

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In addition, we define $0! = 1$.

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Remark

The definition of $0!$ makes a lot of formulas convenient.

Example: compute expressions involving factorial

Example

Compute the following expressions:

(a) $6!$;

(b) $\frac{n!}{(n-2)!}$ *for integer* $n \geq 3$.

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Solution

(a) $6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720.$

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Compute the following expressions:

(a) $6!$;

(b) $\frac{n!}{(n-2)!}$ *for integer* $n \geq 3$.

Solution

(a) $6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$.

(b) $\frac{n!}{(n-2)!} = \frac{n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 2 \cdot 1}{(n-2) \cdot \ldots \cdot 2 \cdot 1} = n(n-1)$.

The "n choose r" symbol

Definition

Let n and r be integers with $0 \le r \le n$. The symbol $\binom{n}{r}$ is read "n choose r" and represents the number of subsets of size r that can be chosen from a set with n elements.

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The formula of "n choose r" involves factorials.

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Proposition

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

Corollary

For integers $0 \leq r \leq n$, $\binom{n}{r} = \binom{n}{n-r}$.

Example: computing $\binom{n}{r}$

Example

Compute the following expressions:

- (a) $\binom{4}{2}$;
- (b) $\binom{n+1}{n}$ *for integer* $n \in \mathbb{N}$.

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(a) $\binom{4}{2} = \frac{4!}{2!2!} = \frac{24}{2 \cdot 2} = 6.$

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Compute the following expressions:

- (a) $\binom{4}{2}$;
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Solution

(a) $\binom{4}{2} = \frac{4!}{2!2!} = \frac{24}{2 \cdot 2} = 6.$

(b) $\binom{n+1}{n} = \frac{(n+1)!}{n!1!} = n + 1.$

Binomial coefficient

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Theorem

Let a and b be variables and $n \in \mathbb{N}$. We have the following identity:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}.$$

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Theorem

Let a and b be variables and $n \in \mathbb{N}$. We have the following identity:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}.$$

Proof.

In order to get a product $a^k b^{n-k}$, we need to choose an a from k parentheses and choose a b from the remaining $n-k$ parentheses. So the total numbers of choices is the number of ways to choose an k-element subset from a set with n elements, which is $\binom{n}{k}$. □

HW#6 of this section

Section 5.1 Exercise 4, 11, 19, 36, 43, 68.