Math 325K - Lecture 12
Section 5.2 Mathematical Induction I

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Outline

- Principle of mathematical induction.
- Sum of terms in special sequences.

Motivation

Suppose we need to check conjectures about repeated procedures involving infinitely many cases, of course we can not work out the cases one at a time. If they have a similar pattern, we may utilize it and follow a shortcut - mathematical induction.

The principle

Definition

Let $P(n)$ be a property that is defined for integers n, and let a be a fixed integer. Suppose the following two statements are true:

1. *$P(a)$ is true.*
2. *For all integers $k \geq a$, if $P(k)$ is true, then $P(k+1)$ is true.*

Then the statement "for all integers $n \geq a$, $P(n)$" is true.

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Remark

We usually choose $a = 1$, while a could be any integer.

Why this principle is true

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Remark

An illustration of this principle is an infinite sequence of dominos positioned one behind the other in such a way that if any given domino falls backward, it makes the one behind it fall backward also. If the very first one falls backward, what would happen? They all fall.

Method of proof

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Remark

Note that the inductive step is a universal statement. So we usually take a particular but arbitrarily chosen integer $k \geq a$ and suppose that $P(k)$ is true (this is called the **inductive hypothesis***). Finally we try to draw the conclusion that $P(k+1)$ is true.*

Example: proof by induction

Example

Show that for every positive integer n*, we have that* $2^n > n$.

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Show that for every positive integer n, we have that $2^n > n$.

Proof.

Let $P(n)$ be the property $2^n > n$. We use induction and let $a = 1$. The basis step is $P(1)$. When $n = 1$, $2^n = 2 > 1 = n$, so $P(1)$ is true. As for the inductive step, suppose $k \geq 1$ is an arbitrary integer and $P(k)$ is true. Then we have $2^k > k$. Now we need to check $P(k+1)$. Note that

$$2^{k+1} = 2 \cdot 2^k = 2^k + 2^k \geq k + 2^1 > k + 1.$$

So $P(k+1)$ is also true and we are done. □

Example: challenges in the inductive step

Example

The currency of some country only has two types of coins - 3 cents and 5 cents. Prove that for integer $n \geq 8$, we can obtain exactly n cents using those coins.

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Hint

$$8 = 5 + 3, 9 = 3 + 3 + 3, 10 = 5 + 5, 11 = 5 + 3 + 3, \cdots$$

Division into cases in the inductive step

Proof.

We use induction on n. The basis step is when $n = 8$. Since $8 = 3 + 5$, it is true. For the inductive step, suppose $k \geq 8$ is an arbitrary integer such that we can obtain k cents using those coins. How to obtain $k + 1$ cents? There are two cases.

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Case 1: at least one 5-cent coin is used. Then we may replace one such coin by two 3-cent coins, so $k + 1$ cents is obtained.

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Case 1: at least one 5-cent coin is used. Then we may replace one such coin by two 3-cent coins, so $k + 1$ cents is obtained.

Case 2: no 5-cent coin is used. Since $k \geq 8$, at least three 3-cent coins are used. We can take three of them, and replace them by two 5-cent coins, so $k + 1$ cents is also obtained.

In summary, $k + 1$ cents is also able to obtain and we are done. □

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Remark

This is a typical example for an alternative form of induction - we will discuss it next week.

Closed form

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Remark

A closed form may be a sum too, but it must contain a constant number of summands.

Arithmetic progressions

Definition

An **arithmetic progression** *is a sequence in which the difference of any two consecutive terms is a constant. In other words, it is a sequence of the form* $a, a+d, a+2d, \ldots$*, where* $a \in \mathbb{R}$ *is called the* **initial term** *and* $d \in \mathbb{R}$ *is called the* **common difference**.

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Remark

The common difference may be zero. Arithmetic progressions may be finite or infinite.

Example: sum of terms in an arithmetic progression

Example

(1) Evaluate $\sum_{k=1}^{100} k$.

(2) Show that $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$ *for all positive integers* n.

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(2) Show that $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$ *for all positive integers* n.

Solution

(1) We can pair up integers from 1 *to* 100*:*
$1 + 100 = 101, 2 + 99 = 101, \cdots$. *There are* $100/2 = 50$ *pairs in total, and the sum of each pair is* 101*, so the total sum is* $101 \cdot 50 = 5050$.

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Remark

There is a story that the preeminent German mathematician Carl Friedrich Gauss managed to quickly get the answer using this method when he was a young child.

Example: sum of terms in an arithmetic progression

Solution

(2) We use induction on n. When $n = 1$, the claim becomes $1 = \frac{1 \cdot 2}{2}$, which is trivially true. For the inductive step, suppose m is an arbitrary positive integer such that the claim is true when $n = m$, then $\sum_{k=1}^{m+1} k = \frac{m(m+1)}{2}$. Now we consider the case when $n = m + 1$. We have

$$\sum_{k=1}^{m+1} k = \sum_{k=1}^{m} k + (m+1)$$

$$= \frac{m(m+1)}{2} + (m+1) = (m+1)(\frac{m}{2} + 1) = \frac{(m+1)(m+2)}{2}.$$

So the claim is still true when $n = m + 1$. We are done.

Sum of terms in general arithmetic progressions

Theorem

For positive integer n and real number d, we have:

$$\sum_{k=0}^{n-1} (a + kd) = na + \frac{n(n-1)}{2} d.$$

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We can prove this theorem in both ways: pairs or induction.

Geometric progressions

Definition

An **geometric progression** *is a sequence in which the ratio of any two consecutive terms is a constant. In other words, it is a sequence of the form* $a, ar, ar^2, \ldots$, *where* $a \in \mathbb{R} - \{0\}$ *is called the* **initial term** *and* $r \in \mathbb{R} - \{0\}$ *is called the* **common ratio**.

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Remark

Be careful that the common ratio cannot be zero! As a corollary, all terms in a geometric progression may not be zero.

Sum of terms in general geometric progressions

Theorem

For positive integer n and real number $r \neq 0$, we have:

$$\sum_{k=0}^{n-1} ar^k = \begin{cases} na, & \text{if } r = 1; \\ a\frac{r^n-1}{r-1}, & \text{if } r \neq 1. \end{cases}$$

Sum of terms in general geometric progressions

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Proof.

The case when $r = 1$ is simple: $ar^k = a$ for all k, so the sum equals to na. Now we assume that $r \neq 1$. Let S be the sum. Then $rS = \sum_{k=0}^{n-1} ar^{k+1} = \sum_{k=1}^{n} ar^k$. So

$$rS - S = ar^n - a,$$

$$S = a\frac{r^n - 1}{r - 1}.$$

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Evaluate the following sum of terms in a geometric sequence:

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Solution

Here we have a geometric sequence with 9 *terms. The initial term is* $2^0 = 1$ *and the common ratio is* 2*. By the theorem, the answer is*

$$1 \cdot \frac{2^9 - 1}{2 - 1} = 512 - 1 = 511.$$

Method: guess first, prove by induction

In many situations, we don't know the answer of a sum. However, if we manage to figure it out, it is trivial to justify it by induction.

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Let n be a positive integer. Evaluate the following sum:

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Hint

The first few terms in the sequence: $1, 4, 18, 96.$

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Hint

The first few terms in the sequence: $1, 4, 18, 96$.
And the corresponding sums for small n: $1, 5, 23, 119$. *Any pattern?*

Method: guess first, prove by induction

Solution

We claim that the sum is $(n+1)! - 1$*, and we use induction to prove it. When* $n = 1$*, the sum is* 1 *and* $(1+1)! - 1 = 2 - 1 = 1$*. As for the inductive step, suppose* m *is an arbitrary positive integer such that* $\sum_{k=1}^{m} k \cdot k! = (m+1)! - 1$*. Then*

$$\begin{aligned}\sum_{k=1}^{m+1} k \cdot k! = \sum_{k=1}^{m} k \cdot k! + (m+1) \cdot (m+1)! \\ = [(m+1)! - 1] + (m+1) \cdot (m+1)! \\ =(m+2) \cdot (m+1)! - 1 = (m+2)! - 1.\end{aligned}$$

So the claim is still true when $n = m+1$*. We are done.*

HW#6 of this section

Section 5.2 Exercise 4, 11, 21, 33.