Math 325K - Lecture 13
Section 5.3 & 5.4
Mathematical Induction II

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Outline

- More applications of mathematical induction.
- The principle of strong mathematical induction.
- Common mistakes when using induction.

Application: divisibility

Remark

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More specific, if the inductive hypothesis is $a \mid b$, and we need to show that $a \mid c$ in the inductive step, there are two common methods:

- show that $a \mid (c-b)$;
- show that c is a multiple of b.

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Example

Show that for every positive integer n, we have that $n^3 - n$ is divisible by 6.

Example: show divisibility using induction

Proof.

We use mathematical induction. The basis step is when $n = 1$. We have $n^3 - n = 0$, so it is divisible by 6.

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As for the inductive step, suppose k is an arbitrary positive integer such that the claim is true when $n = k$. We have an inductive hypothesis $6 \mid (k^3 - k)$. Now we need to consider the case when $n = k + 1$. In other words, we need to check whether 6 divides $(k+1)^3 - (k+1)$. Of course we would like to link this number with $k^3 - k$ in the inductive hypothesis. We get

$$(k+1)^3 - (k+1) = k^3 + 3k^2 + 3k + 1 - k - 1 = \left(k^3 - k\right) + 3k^2 + 3k.$$

Now it suffices to show that 6 divides $3k(k+1)$. Note that $k(k+1)$ must be even, so there exists an integer m such that $k(k+1) = 2m$, then $3k(k+1) = 6m$ is a multiple of 6, the inductive step is done. $\square$

Application: inequality

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Let's see a challenging example.

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$$\sqrt{n} < \sum_{k=1}^{n} \frac{1}{\sqrt{k}}.$$

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Example

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$$\sqrt{n} < \sum_{k=1}^{n} \frac{1}{\sqrt{k}}.$$

Remark

Why do we specify $n \geq 2$*? because the claim is false when* $n = 1$.

Example: show inequality using induction

Proof.

We use mathematical induction. The basis step is when $n = 2$. We have

$$\sqrt{n} = \sqrt{2}, \sum_{k=1}^{n} \frac{1}{\sqrt{k}} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} = 1 + \frac{\sqrt{2}}{2}.$$

Since $\sqrt{2} < \sqrt{4} = 2$, we have $\frac{\sqrt{2}}{2} < 1$, and thus

$$\sqrt{2} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} < 1 + \frac{\sqrt{2}}{2}.$$

Basis step is done. (to be continued) □

Example: show inequality using induction

Proof.

(Continued) For the inductive step, let $m \geq 2$ be an arbitrary integer such that the claim is true for $n = m$, then we have the inductive hypothesis

$$\sqrt{m} < \sum_{k=1}^{m} \frac{1}{\sqrt{k}}.$$

We need to show the case when $n = m + 1$, which is

$$\sqrt{m+1} < \sum_{k=1}^{m+1} \frac{1}{\sqrt{k}}.$$

(to be continued) □

Example: show inequality using induction

Proof.

(Continued) Now we compare the two inequalities. Note that

$$\sum_{k=1}^{m+1} \frac{1}{\sqrt{k}} = \sum_{k=1}^{m} \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{m+1}} > \sqrt{m} + \frac{1}{\sqrt{m+1}}.$$

It suffices to show that

$$\sqrt{m} + \frac{1}{\sqrt{m+1}} > \sqrt{m+1}.$$

(to be continued) □

Example: show inequality using induction

Proof.

(Continued) For this final step, there are multiple approaches. For example, it is equivalent to show that

$$\frac{1}{\sqrt{m+1}} > \sqrt{m+1} - \sqrt{m}.$$

Since $(m+1) - m = 1$ and $x^2 - y^2 = (x+y)(x-y)$, we have

$$\left(\sqrt{m+1} + \sqrt{m}\right) \cdot \left(\sqrt{m+1} - \sqrt{m}\right) = 1.$$

By rationalizing the numerator this time, we get

$$\sqrt{m+1} - \sqrt{m} = \frac{1}{\sqrt{m+1} + \sqrt{m}} < \frac{1}{\sqrt{m+1}}.$$

□

Motivation: stronger inductive hypothesis

Recall our example of 3- and 5-cents coins. When carrying out the inductive step, it is natural to add a coin, but then the total value of the coins would increase by 3 instead of 1. Then can we modify the principle to deal with this situation?

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Recall our example of 3- and 5-cents coins. When carrying out the inductive step, it is natural to add a coin, but then the total value of the coins would increase by 3 instead of 1. Then can we modify the principle to deal with this situation?

Remark

If we think carefully about the pattern, we note that the inductive hypothesis is usually "$P(k)$ is true" for some arbitrary positive integer k. Suppose we begin with $P(1)$. When we already get $P(k)$, we must have got all of $P(1), P(2), P(3), \cdots, P(k-1)$! So it is possible to strengthen the inductive hypothesis.

Principle of strong mathematical induction

Definition

The principle of **strong mathematical induction** *is the following: Let $P(n)$ be a property that is defined for integers n, and let a and b be fixed integers with $a \leq b$. Suppose the following two statements are true:*

1. *(**basis step**)* $P(a), P(a+1), \cdots, P(b)$ *are all true.*
2. *(**inductive step**) For any integer $k \geq b$, if $P(i)$ is true for all integers $a \leq i \leq k$, then $P(k+1)$ is true.*

Then the statement "for all integers $n \geq a$, $P(n)$" is true. The supposition that $P(i)$ is true for all integers $a \leq i \leq k$ is called the **inductive hypothesis**.

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Then the statement "for all integers $n \geq a$, $P(n)$" is true. The supposition that $P(i)$ is true for all integers $a \leq i \leq k$ is called the **inductive hypothesis**.

Example: divisible by a prime

Let's redo the proof of this theorem:

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Show that any integer n greater than 1 is divisible by a prime number.

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Basis step: $n = 2$ is divisible by the prime number 2.

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Example

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Proof.

Basis step: $n = 2$ is divisible by the prime number 2.
Inductive step: suppose $k \geq 2$ is an integer such that for integers i with $2 \leq i \leq k$, i is divisible by a prime number. Now we consider the case when $n = k + 1$. To analyze the divisors of $k + 1$, we need to know whether it is prime or composite (must be in either case as it is greater than 1). We apply division into cases. (to be continued) □

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Proof.

(Continued) Case 1: $k+1$ is a prime number. Then it is divisible by itself which is a prime number.

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Case 2: $k+1$ is a composite number. Then there exist positive integers $a, b > 1$ such that $k+1 = ab$. Since $b > 1$, a must be less than $k+1$. So $2 \leq a \leq k$. We apply the inductive hypothesis, then a is divisible by a prime number p. By the transitivity of divisibility, p also divides $k+1$. So our inductive step is done. $\square$

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Proof.

(Continued) Case 1: $k+1$ is a prime number. Then it is divisible by itself which is a prime number.
Case 2: $k+1$ is a composite number. Then there exist positive integers $a, b > 1$ such that $k+1 = ab$. Since $b > 1$, a must be less than $k+1$. So $2 \leq a \leq k$. We apply the inductive hypothesis, then a is divisible by a prime number p. By the transitivity of divisibility, p also divides $k+1$. So our inductive step is done. □

Remark

If we don't which known case we need as a premise to deduce the next case, we may apply the strong mathematical induction.

Example: a flawed proof using induction

Example

Here is a "proof" of the false statement "for all integers $n \geq 1$, $3^n - 2$ is even."

Proof.

Suppose the statement is true for an arbitrary integer $k \geq 1$. Then $3^k - 2$ is even. We must show that $3^{k+1} - 2$ is even. But

$$3^{k+1} - 2 = 3^k \cdot 3 - 2 = (3^k - 2) + 2 \cdot 3^k.$$

Now $3^k - 2$ is even by inductive hypothesis and $2 \cdot 3^k$ is even by definition. Hence their sum is also even. It follows that $3^{k+1} - 2$ is even, which is what we needed to show. □

What is the flaw?

Induction makes no sense without basis step

Solution

All steps in the "proof" are correct, but it misses the basis step and in fact the basis step is apparently false.

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Remark

Although the inductive step is usually the essential step in a proof by induction, please note that it is only a conditional statement! So if the premise is false, it is an unsound argument and it is useless. As a result, it is vital to make sure that the basis step is done correctly.

Example: a hidden flaw

Example

Here is a "proof" of the false statement "for all nonzero real numbers r and nonnegative integer n, $r^n = 1$."

Proof.

Fix r, we use strong induction on n. Basis step: when $n = 0$, since $r \neq 0$, $r^0 = 1$ is true.
Inductive step: suppose $k \geq 0$ is an arbitrary integer such that $r^i = 1$ for all $0 \leq i \leq k$. Note that $r^{k+1} = r^{k+k-(k-1)} = r^k \cdot r^k / r^{k-1}$. By the inductive hypothesis, $r^k = r^{k-1} = 1$, so r^{k+1} is also 1. The inductive step is done. □

What is the flaw?

Mind the range of numbers that the hypothesis applies to

Solution

The basis step is absolutely correct. In the inductive step, the formulas are correct. The flaw is that when $k = 0$, $k - 1 = -1$, *which is no longer between* 0 *and* k*! So in this particular case we don't have* $r^{k-1} = 1$*!*

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Remark

In this flawed proof, we can see that if we already have that the claim is true for $k = 0, 1$, *then the proof works. But this is expected - when* $k = 1$, *the claim becomes* $r = 1$, *and if* $r = 1$, *the statement would be true. But it's false otherwise.*

HW# 7 of today's sections

Section 5.3 Exercise 7, 10, 30.