Math 325K - Lecture 14
Section 5.4 & 5.5

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Outline

- The well-ordering principle for integers.
- Recurrence relations.
- Application of strong mathematical induction - sequences.

The principle

Axiom

Let S be a nonempty set of integers all of which are greater than some fixed integer C. Then S has a least element.

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Remark

This principle is equivalent to the principle of mathematical induction. In other words, either one could imply the other.

Examples: finding least elements

Example

For the following sets, if the set has a least element, find it. Otherwise explain why the well-ordering principle is not violated.

(a) $\{16 - 3k \in \mathbb{N} \mid k \in \mathbb{Z}\}$.

(b) $\{x \in \mathbb{Q} \mid x > 0\}$.

(c) $\{n \in \mathbb{N} \mid n > n^2\}$.

Examples: finding least elements

Solution

(a) If $16 - 3k \in \mathbb{N}$, *then* $k < \frac{16}{3} = 5 + \frac{1}{3}$. *Since* $k \in \mathbb{Z}$, k *is at most* 5, *so* $16 - 3k$ *is at least* $16 - 3 \cdot 5 = 1$.

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Solution

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(b) Suppose $y = \frac{p}{q}$ *is the least element in this set, then its half,* $\frac{p}{2q}$ *that is even smaller is also in the set, a contradiction! So there is no least element. But the well-ordering principle is only for sets of integers, so it does not apply to rational numbers.*

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(c) Suppose $n \in \mathbb{N}$*, then* $n \geq 1$*, which implies that* $n^2 = n \cdot n \geq n \cdot 1 = n$*. So the set is empty and it does not have a least element.*

Application: existence part of quotient-remainder theorem

Theorem

Given $n \in \mathbb{Z}$ *and* $d \in \mathbb{N}$*, there exists integers* q *and* r *such that* $n = dq + r$ *and* $0 \leq r < d$*.*

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Theorem

Given $n \in \mathbb{Z}$ and $d \in \mathbb{N}$, there exists integers q and r such that $n = dq + r$ and $0 \leq r < d$.

Proof.

Let S be the set $\{n - dk \mid k \in \mathbb{Z}, n - dk \geq 0\}$. We claim that S is nonempty. Because if we choose $k = -|n|$, then $n - d(-|n|) = n + d|n| \geq n + |n| \geq 0$, so it belongs to S. By the well-ordering principle for the integers, S contains a least element r (then $r \geq 0$), and there exists an integer q such that $r = n - dq$. Now we consider another number $n - d(q+1) = n - dq - d = r - d$. Since $r - d$ is strictly smaller than r, it cannot belong to S. While $q + 1 \in \mathbb{Z}$, so it must be the case that $r - d \geq 0$ does not hold. Hence $r - d < 0$, $r < d$. □

Application: proving existential statements/proof by contradiction

Remark

The well-ordering principle for the integers is also frequently used in combination with proof by contradiction. The pattern is the following:

- (a) *In order to justify that a statement p is true, we assume that p is false.*
- (b) *Next we construct some nonempty set S of integers that have a lower bound.*
- (c) *Then by the well-ordering principle for the integers, we can choose the least element r of S.*
- (d) *With the assumption that p is false, we can find another element in S that is smaller than r, which is the desired contradiction.*

The prime divisor example revisited

Example

Using the well-ordering principle for the integers to show that any integer n greater than 1 is divisible by a prime number.

The prime divisor example revisited

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Proof.

Let S be the set of all positive integers at least 2 that is not divisible by any prime number. Suppose S is nonempty, then by the well-ordering principle for the integers, S has a least element $r \geq 2$. Since $r > 1$, it is either prime or composite. If r is prime, then it is divisible by itself which is a prime number, a contradiction to the fact that $r \in S$; if r is composite, then there exist integers $a, b > 1$ such that $r = ab$. Since $b > 1$, a must be strictly less than r. Then $a \notin S$. Note that $a \geq 2$, so it must be the case that a is divisible by some prime number, say p. So we have $p \mid a$ and $a \mid r$, then $p \mid r$, a contradiction to the fact that $r \in S$. So our assumption is false and $S = \emptyset$. □

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A **recurrence relation** *for a sequence* $a_0, a_1, a_2, \cdots$ *is a formula that relates each term* a_k *to certain of its predecessors* $a_{k-1}, a_{k-2}, ..., a_{k-i}$, *where* i *is an integer with* $k - i \geq 0$. *The initial conditions for such a recurrence relation specify the values of* $a_0, a_1, a_2, \cdots, a_{m-1}$, *where* m *is* i *or some other positive integer. The sequence* $\{a_n\}$ *is also called* **recursively defined**.

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Remark

The way we define a recurrence relation is very similar to the principle of strong mathematical induction.

Examples: computing terms in recursively defined sequences

Example

Suppose $\{a_n\}_{n\geq 0}$ *is a sequence with* $a_0 = 0$ *and* $a_1 = 1$.

1. (a) *If* $a_n = 2a_{n-1} - a_{n-2}$ *for integers* $n \geq 2$, *evaluate* a_4 *and* a_5.
2. (b) *If* $a_n = a_{n-1} + a_{n-2}$ *for integers* $n \geq 2$, *evaluate* a_4 *and* a_5.

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Solution

(a) $a_2 = 2a_1 - a_0 = 2, a_3 = 3, a_4 = 4, a_5 = 5$.

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Solution

(a) $a_2 = 2a_1 - a_0 = 2, a_3 = 3, a_4 = 4, a_5 = 5$.
(b)
$a_2 = a_1 + a_0 = 1, a_3 = 1 + 1 = 2, a_4 = 2 + 1 = 3, a_5 = 3 + 2 = 5$.

Example: Fibonacci number

Fibonacci is a great Italian mathematician in the 13th century. He considered the rapid reproduction of rabbits and introduced the following sequence:

Definition

The Fibonacci sequence F_n is defined as follows:

- $F_0 = 0, F_1 = 1$;
- $F_n = F_{n-1} + F_{n-2}$ *for all integers* $n \geq 2$.

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Remark

Fibonacci numbers have a lot of properties, and itself even became a small branch of mathematical research (there are research journals about them).

An explicit formula for F_n

It is quite obvious that F_n are all nonnegative integers and they are increasing with respect to n. Do we have an explicit formula?

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Theorem

For all integers $n \geq 0$, we have

$$F_n = \frac{1}{\sqrt{5}} \cdot \left[\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right].$$

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Remark

Since $\left| \frac{1-\sqrt{5}}{2} \right| < 1$, asymptotically F_n is like a geometric progression with common ratio $\frac{1+\sqrt{5}}{2} \approx 1.618$.

Example: compound interests

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Suppose you deposit $10,000$ *dollars into a bank account that earns* 4% *interest compounded annually. Let* A_n *be the balance in your account after* n *years. Find an explicit formula of* A_n.

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Solution

After $n \geq 0$ *years, the account has a balance of* A_n*. In the next year, the interest is* 4%*, which equals to* $A_n \cdot 4\%$*. So the total balance after next year would be* $A_n \cdot (1 + 4\%) = 1.04 \cdot A_n$*. Hence* A_n *is recursively defined as*

$$A_{n+1} = 1.04 \cdot A_n.$$

So A_n *is a geometric progression and* $A_n = A_0 \cdot 1.04^n = 10000 \cdot 1.04^n$*.*

Why we need strong mathematical induction

Suppose we would like to prove a property of the terms in a recursively defined sequence. For example, $F_n \in \mathbb{Z}$ for all $n \geq 0$. It is natural to apply mathematical induction. While if we apply the ordinary version, we may face the following issue:

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In the inductive step, suppose $k \geq 1$ is an arbitrary integer such that $F_k \in \mathbb{Z}$. We need to consider F_{k+1}. By the recurrence relation, $F_{k+1} = F_k + F_{k-1}$. We only know that $F_k \in \mathbb{Z}$, but what about F_{k-1}? It is not addressed in the inductive hypothesis! So the inductive step in this version of mathematical induction cannot proceed.

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But if we apply the strong mathematical induction instead, it becomes a piece of cake: by the stronger inductive hypothesis, since $0 \leq k-1 \leq k$, $F_{k-1} \in \mathbb{Z}$ also holds. Then F_{k+1} is the sum of two integers, which is still an integer and the inductive step is done.

Example: proving explicit formula

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Suppose $\{a_n\}_{n\geq 0}$ *is a sequence with* $a_0 = 0, a_1 = 1$ *and* $a_n = 2a_{n-1} - a_{n-2}$ *for integers* $n \geq 2$*. Show that* $a_n = n$ *for all integers* $n \geq 0$*.*

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Proof.

We use strong mathematical induction on n. Basis step: the claim is true for $n = 0, 1$. Suppose $k \geq 1$ is an arbitrary integer such that the claim is true for integers i with $0 \leq i \leq k$. Now we consider the case when $n = k + 1$.

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$$a_{k+1} = 2a_k - a_{k-1} = 2k - (k - 1) = 2k - k + 1 = k + 1.$$

The inductive step is done. □

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$$F_{k+1} = F_k + F_{k-1} < 2^k + 2^{k-1} = 3 \cdot 2^{k-1} < 4 \cdot 2^{k-1} = 2^{k+1}.$$

The inductive step is done. □

HW #7 of today's sections

Section 5.4 Exercise 7, 11, 21, 26.

Section 5.5 Exercise 6, 14, 32.