Math 325K - Lecture 15
Section 6.1 Set Theory

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October 23rd, 2018

Outline

- Subsets and the element method of proof.
- Venn diagrams.
- Operations on sets.

Subsets and proper subsets

Recall we introduced the notions of "sets", which cannot be defined.

Definition

Let A and B be two sets. A is called a **subset** *of B if every element of A is also an element of B, and denoted by $A \subseteq B$. Formally,*

$$A \subseteq B \Leftrightarrow \forall x, (x \in A) \rightarrow (x \in B).$$

In addition, A is called a **proper subset** *of B, denoted by $A \subsetneq B$, if A is a subset of B and $A \neq B$. In other words, other than $A \subseteq B$, $\exists y \in B$ such that $y \notin A$.*

Example: follow the definition

Example

Let $A = \{1\}$ *and* $B = \{1, \{1\}\}$. *(a) Is* $A \subseteq B$ *? (b) If yes, then is* A *a proper subset of* B*?*

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Solution

(a) Since A *has only one element* 1, *we just need to check whether it belongs to* B. *Actually it is, so the universal statement* $A \subseteq B$ *is true.*

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Solution

(a) Since A has only one element 1, we just need to check whether it belongs to B. Actually it is, so the universal statement $A \subseteq B$ is true.
(b) Note that $\{1\}$ is an element in B, but it is not in A, so A is a proper subset of B.

Example: justify subsets

Example

Let $A = \{6k + 5 \mid k \in \mathbb{Z}\}$ *and* $B = \{3r + 2 \mid r \in \mathbb{Z}\}$. *Show that* $A \subseteq B$.

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Proof.

It suffices to show that for every element $x \in A$, we also have $x \in B$. Suppose x is an arbitrary element of A. Then there exists an integer k such that $x = 6k + 5$.

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Proof.

It suffices to show that for every element $x \in A$, we also have $x \in B$. Suppose x is an arbitrary element of A. Then there exists an integer k such that $x = 6k + 5$. Now we need to show that $x \in B$, so we need to find an integer r such that x is also equal to $3r + 2$. Note that $6k + 5 = (6k + 3) + 2 = 3 \cdot (2k + 1) + 2$, so we can choose $r = 2k + 1$. Since k is an integer, so is $2k + 1$. By definition, $x \in B$ and we are done. $\square$

Equality of sets

Two sets are equal if and only if they have exactly the same elements. In other words,

Definition

Let A and B be two sets. They are called **equal** *if and only if every element of A belongs to B and every element of B belongs to A. Formally,*

$$A = B \Leftrightarrow A \subseteq B \wedge B \subseteq A.$$

Definitions

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- *The* **difference** *of B minus A, denoted $B - A$, is the set of all elements that are in B and not in A.*
- *The* **complement** *of A, denoted A^c, is the set of all elements in U that are not in A.*

Formal definition

Remark

In symbols, we have

$$A \cup B = \{x \in U \mid x \in A \text{ or } x \in B\}.$$

$$A \cap B = \{x \in U \mid x \in A, x \in B\}.$$

$$B - A = \{x \in B \mid x \notin A\}.$$

$$A^c = \{x \in U \mid x \notin A\} = U - A.$$

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Remark

The complement is a little special that we need to fix a universal set before talking about it.

Example: operations

Example

Let $U = \{1, 2, 3, 4, 5\}$, $A = \{2, 4\}$ *and* $B = \{2, 3, 5\}$.

- (a) *Find* $A \cup B$.
- (b) *Find* $B - A$.
- (c) *Find* B^c.
- (d) *Find* $(A \cap B)^c$.

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Solution

(a) $A \cup B = \{2, 3, 4, 5\}$.

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Solution

(a) $A \cup B = \{2, 3, 4, 5\}$. *(b)* $B - A = \{3, 5\}$.

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- (a) *Find* $A \cup B$.
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- (d) *Find* $(A \cap B)^c$.

Solution

(a) $A \cup B = \{2, 3, 4, 5\}$. *(b)* $B - A = \{3, 5\}$. *(c)* $B^c = \{1, 4\}$. *(d)* $A \cap B = \{2\}$, *so* $(A \cap B)^c = \{1, 3, 4, 5\}$.

Empty set

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Remark

For all set A, $\emptyset \subseteq A$. $\emptyset$ is very special that it could make a lot of statements false. For example the well-ordering principle for the integers. So when checking the truth value of statements, do not forget this set!

Disjoint sets

Definition

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We can generalize this definition.

Definition

Sets $A_1, A_2, A_3, \ldots$ are called **mutually disjoint** *(or* **pairwise disjoint***) if and only if for all $1 \leq i < j$, $A_i \cap A_j = \emptyset$.*

Partitions of sets

Definition

A finite or infinite collection of nonempty sets $\{A_1, A_2, A_3, \ldots\}$ *is a* **partition** *of a set* A *if and only if*

1. A *is the union of all the* A_i.
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Example

The set of even integers and the set of odd integers form a partition of $\mathbb{Z}$.
$\mathbb{Q}$ *and the set of all irrational numbers form a partition of* $\mathbb{R}$.

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Given a set A, the **power set** *of A, denoted $\mathscr{P}(A)$, is the set of all subsets of A.*

Remark

The **power set axiom** *guarantees that $\mathscr{P}(A)$ is always a set. $\emptyset$ belongs to the power set of any set. When A is infinite, $\mathscr{P}(A)$ is infinite too.*

Example: find the power set

Example

Let $A = \{0, \{0\}\}$. *How many elements are there in* A? *What is* $\mathscr{P}(A)$?

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Let $A = \{0, \{0\}\}$. *How many elements are there in* A? *What is* $\mathscr{P}(A)$?

Solution

There are 2 *elements in* A: 0 *and* $\{0\}$.

Example: find the power set

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Let $A = \{0, \{0\}\}$*. How many elements are there in* A*? What is* $\mathscr{P}(A)$*?*

Solution

There are 2 *elements in* A*:* 0 *and* $\{0\}$*.*

$$\mathscr{P}(A) = \{\emptyset, \{0\}, \{\{0\}\}, \{0, \{0\}\}\}.$$

Motivation

Recall that we used diagrams to test for validity of arguments. This methods applies to sets too, and it becomes more rigorous.

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Recall that we used diagrams to test for validity of arguments. This methods applies to sets too, and it becomes more rigorous. If sets A and B are represented as regions in the plane, relationships between A and B can be represented by pictures, called **Venn diagrams**, that were introduced by the British mathematician John Venn in 1881.

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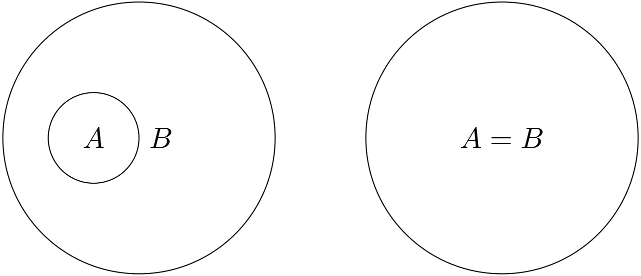

Figure: Venn diagrams for $A \subseteq B$

Diagram of other related sets

The following pictures shows union, intersection and complement of sets.

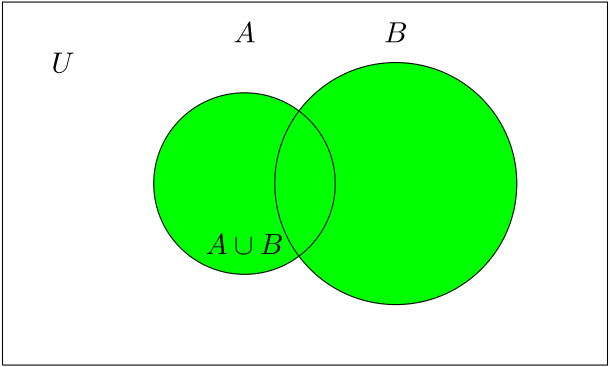

Figure: The set $A \cup B$.

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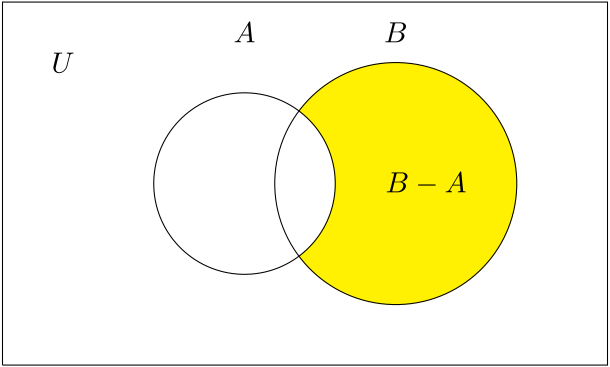

Figure: The set $B - A$.

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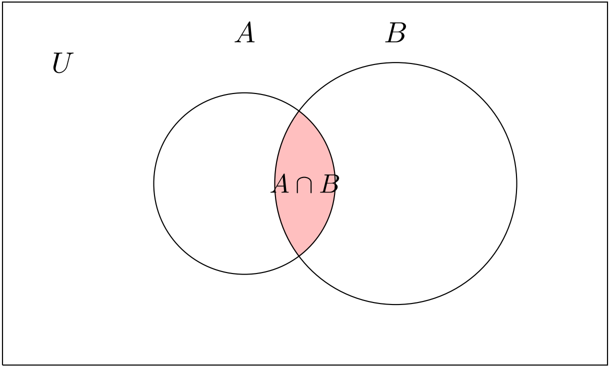

Figure: The set $A \cap B$.

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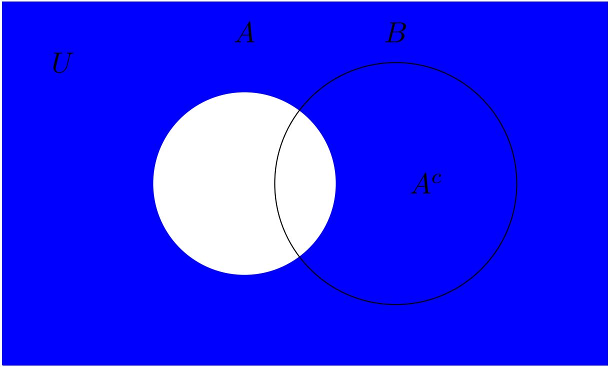

Figure: The set A^c.

Example: diagram for disjoint subsets

Example

Let U be the universal set and A, B be subsets of U. If A and B are disjoint, draw a Venn diagram including A, B, U to illustrate this relationship.

Example: diagram for disjoint subsets

Solution

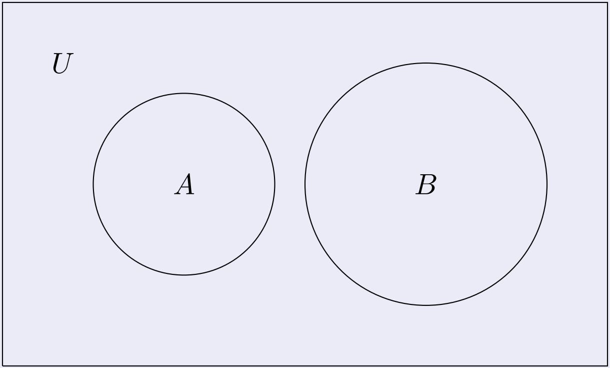

Figure: *The disjoint subsets A and B of U.*

HW #8 of this section

Section 6.1 Exercise 1(b)(d), 7, 10(f)(g)(h), 14(b), 20, 30.