Math 325K - Lecture 16
Section 6.2 Properties of Sets

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Outline

- Set properties.
- Set identities.
- Proof techniques related to sets.

Some subset relations

Theorem

For all sets A, B, C, we have the following properties:

(a) $A \cap B \subseteq A, A \cap B \subseteq B$;

(b) $A \subseteq A \cup B, B \subseteq A \cup B$;

(c) *if* $A \subseteq B$ *and* $B \subseteq C$, *then* $A \subseteq C$.

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Remark

There are still analogues: $\wedge$ *with* $\cap$; *and* $\vee$ *with* $\cup$.

Example of a proof

Here we prove the statement "$B \subseteq A \cup B$."

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Proof.

We need to justify the following universal conditional statement: "for any x, if $x \in B$, then $x \in A \cup B$."

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Here we prove the statement "$B \subseteq A \cup B$."

Proof.

We need to justify the following universal conditional statement: "for any x, if $x \in B$, then $x \in A \cup B$." Let y be an arbitrary element such that $y \in B$. Then y belongs to at least one of A and B, by definition of union, $y \in A \cup B$. $\square$

Example: showing containment

Example

Let A, B, C be sets. Prove that if $A \subseteq B$ and $A \subseteq C$, then $A \subseteq (B \cap C)$.

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Proof.

For any element $x \in A$, since $A \subseteq B$, we have that $x \in B$; since $A \subseteq C$, we have that $x \in C$. Hence $x \in B \cap C$. Therefore $A \subseteq (B \cap C)$. $\square$

More properties

Theorem

Let A, B, C be sets. We have the following properties:

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Remark

We can easily prove them by element method.

The identities

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- *(Idempotent laws)* $A \cup A = A \cap A = A$.

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- *(Idempotent laws)* $A \cup A = A \cap A = A$.
- *(Absorption laws)* $A \cup (A \cap B) = A \cap (A \cup B) = A$.

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If in addition U is the universe set that contains A, B, then we have the following identities:

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- *(Universal bound laws)* $A \cup U = U$, $A \cap \emptyset = \emptyset$.

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- *(Complement of U and $\emptyset$)* $U^c = \emptyset$, $\emptyset^c = U$.

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- *(Universal bound laws)* $A \cup U = U$, $A \cap \emptyset = \emptyset$.
- *(Complement of U and $\emptyset$)* $U^c = \emptyset$, $\emptyset^c = U$.
- *(Set difference laws)* $A - B = A \cap B^c$.

De Morgan's Laws

Recall what we have learned about conjunction and disjunctions. Here in set theory, we have a very similar counterpart:

Theorem

For all sets A and B, we have that

(a) $(A \cup B)^c = A^c \cap B^c$;
(b) $(A \cap B)^c = A^c \cup B^c$.

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b. $(A \cap B)^c = A^c \cup B^c$.

Proof.

Element method for (a): for any element $x \in U$, if x belongs to $(A \cup B)^c$, then x does not belong to $A \cup B$. By definition of union, $x \notin A$ and $x \notin B$, hence $x \in A^c$ and $x \in B^c$. So $x \in A^c \cap B^c$.

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Element method for (a): for any element $x \in U$, if x belongs to $(A \cup B)^c$, then x does not belong to $A \cup B$. By definition of union, $x \notin A$ and $x \notin B$, hence $x \in A^c$ and $x \in B^c$. So $x \in A^c \cap B^c$. Conversely, if $x \in A^c \cap B^c$, then $x \in A^c$ and $x \in B^c$. By definition of complement, $x \notin A$ and $x \notin B$, hence $x \notin A \cup B$ and $x \in (A \cup B)^c$. The two sets contain each other and are equal. $\square$

Example: showing identities

Example

Show that for all sets A, B, C, $(A - B) \cup (C - B) = (A \cup C) - B$.

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Proof.

Let U contain $A \cup B \cup C$ be a universal set. By the set difference law,

$$A - B = A \cap B^c, C - B = C \cap B^c.$$

In addition,

$$(A \cup C) - B = (A \cup C) \cap B^c.$$

Then the claim follows from the distributive law:

$$\begin{aligned}(A - B) \cup (C - B) = (A \cap B^c) \cup (C \cap B^c) = (A \cup C) \cap B^c \\ = (A \cup C) - B.\end{aligned}$$

Generalized distributive law

Theorem

For all sets A and $B_1, B_2, \ldots, B_n$, we have that

(a)
$$A \cup \left(\bigcap_{i=1}^{n} B_i \right) = \bigcap_{i=1}^{n} (A \cup B_i);$$

(b)
$$A \cap \left(\bigcup_{i=1}^{n} B_i \right) = \bigcup_{i=1}^{n} (A \cap B_i).$$

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(b) $$A \cap \left(\bigcup_{i=1}^{n} B_i \right) = \bigcup_{i=1}^{n} (A \cap B_i).$$

Remark

The element method of proof can be applied in the same way as $n = 1$.

Showing set equality

Given sets X and Y. To prove that $X = Y$, one approach is the following

1. Prove that $X \subseteq Y$.
2. Prove that $Y \subseteq X$.

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Remark

To prove containment, we can use element method, or known results.

Showing empty set

There are various ways to show that a given set E is $\emptyset$.

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There are various ways to show that a given set E is $\emptyset$.

- Proof by contradiction. Suppose there is an element $x \in E$, and then deduce a contradiction.
- Show that $E \subseteq \emptyset$.

Example: empty set

Example

Prove that for all sets A, B, C *of the universal set* U*, if* $A \subseteq B$ *and* $B \subseteq C^c$*, then* $A \cap C = \emptyset$.

Example: empty set

Example

Prove that for all sets A, B, C of the universal set U, if $A \subseteq B$ and $B \subseteq C^c$, then $A \cap C = \emptyset$.

Proof.

Proof by contradiction: suppose $x \in A \cap C$. Since $A \subseteq B$, then $x \in B$; since $B \subseteq C^c$, $x \in C^c$, hence $x \notin C$, a contradiction! So $A \cap C = \emptyset$.

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Example

Prove that for all sets A, B, C of the universal set U, if $A \subseteq B$ and $B \subseteq C^c$, then $A \cap C = \emptyset$.

Proof.

Proof by contradiction: suppose $x \in A \cap C$. Since $A \subseteq B$, then $x \in B$; since $B \subseteq C^c$, $x \in C^c$, hence $x \notin C$, a contradiction! So $A \cap C = \emptyset$.

Alternative: By transitivity, $A \subseteq C^c$. By the property we introduced today, $A \cap C \subseteq C^c \cap C$. By complement laws, $C^c \cap C = \emptyset$, so $A \cap C \subseteq \emptyset$, itself is also the empty set. $\square$

Disproving universal statements

To disprove a universal statement, one can present a counterexample. We can either manually present the sets, or illustrate the counterexample by Venn diagram.

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Example

Disprove the statement "for all sets A, B, C, $(A - B) \cup (B - C) = (A - C)$."

How to construct a counterexample?

A counterexample

One simple idea is that in order to make the two sets not equal, we simply need an element that belongs to the first one but does not belong to the second one. In this case, we may choose A, B, C such that an element in $A - B$ is not in $A - C$.

A counterexample

One simple idea is that in order to make the two sets not equal, we simply need an element that belongs to the first one but does not belong to the second one. In this case, we may choose A, B, C such that an element in $A - B$ is not in $A - C$.

Proof.

One counterexample would be $A = C = \{0\}, B = \emptyset$. Then $A - C = \emptyset$ while $(A - B) \cup (B - C) = \{0\}$. □

Number of elements in the power set

Theorem

Let A be a set with n elements, where n is a nonnegative integer. Then $\mathscr{P}(A)$ has 2^n elements.

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Proof.

It suffices to count how many subsets of A are there in total. Note that each set is uniquely determined by its elements. For each subset S of A, its elements are among the n elements of A. For each element of A, it may belong to S or not, so there are two possibilities. And all these elements are independent (they don't affect others), so in total there are 2^n possibilities. $\square$

HW#8 in this section

Section 6.2 Exercise 2(b), 10, 22, 32.