Math 325K - Lecture 17
Section 7.1 Functions defined on general sets

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Outline

- Definitions.
- Properties of functions.
- Examples of functions.

Definition of functions

Recall the definition of functions in Chapter 1:

Definition

A **function** f *from a set* X *to a set* Y*, denoted* $f : X \to Y$*, is a relation from the* **domain** X *to the* **co-domain** Y *such that every element in* X *is related to a unique element in* Y*. If we call this element* y*, then we say that "*f *sends* x *to* y*" or "*f *maps* x *to* y*", and write* $x \xrightarrow{f} y$ *or* $f : x \to y$*. The unique element to which* f *sends* x *is denoted* $f(x)$ *and called "*f *of* x*" or "the value of* f *at* x*".*

Range and preimage

Definition

For a function $f : X \to Y$, the **range** *of f is the set*

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Definition

For a function $f : X \to Y$ *and any* $y \in Y$, *the* **preimage** *of* y, *denoted* $f^{-1}(y)$, *is the set*

$$\{x \in X \mid f(x) = y\}.$$

Functions acting on sets

Definition

If $f : X \to Y$ is a function and $A \subseteq X$ and $C \subseteq Y$, then

$$f(A) = \{y \in Y \mid y = f(x) \text{ for some } x \in A\}.$$

and

$$f^{-1}(C) = \{x \in X \mid f(x) \in C\}$$

$f(A)$ is called the **image** *of A, and $f^{-1}(C)$ is called the* **inverse image** *of C.*

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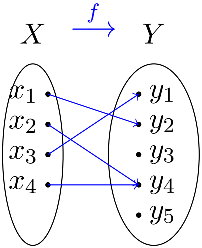

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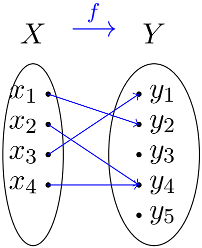
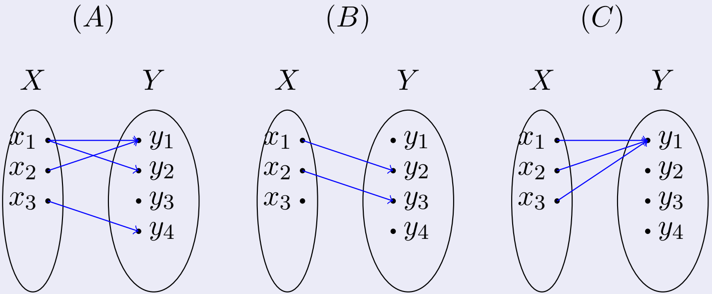


In this example, $f(x_1) = y_2, f(x_2) = f(x_4) = y_4, f(x_3) = y_1$.

Example: arrow diagram

Example

Which of the following arrow diagrams correspond to a function?



Example: arrow diagram

Solution

(A) *does not correspond a function because* $x_1 \in X$ *is related to* 2 *different element s in* Y.

Example: arrow diagram

Solution

(A) does not correspond a function because $x_1 \in X$ is related to 2 different element s in Y.
(B) does not correspond a function because $x_3 \in X$ is not related to any element in Y.

Example: arrow diagram

Solution

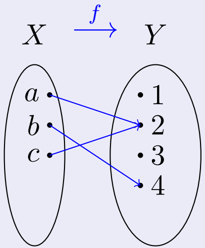
(A) *does not correspond a function because* $x_1 \in X$ *is related to* 2 *different element s in* Y.
(B) *does not correspond a function because* $x_3 \in X$ *is not related to any element in* Y.
(C) *corresponds to a function from* X *to* Y.

Example: read information from arrow diagrams

Example

Let $X = \{a, b, c\}$ *and* $Y = \{1, 2, 3, 4\}$*. Consider the function* $f : X \to Y$ *given by the following arrow diagram.*

- (a) *What is* $f(b)$*?*
- (b) *What is the range of* f*?*
- (c) *What is* $f^{-1}(2)$*?*
- (d) *What is* $f^{-1}(1)$*?*
- (e) *What is* $f(\{a, c\})$*?*



Example: read information from arrow diagrams

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(a) $f(b) = 4$.
(b) The range of f *is* $\{2, 4\}$.
(c) $f^{-1}(2) = \{a, c\}$.
(d) $f^{-1}(1) = \emptyset$.

Example: read information from arrow diagrams

Solution

(a) $f(b) = 4$.
(b) The range of f *is* $\{2, 4\}$.
(c) $f^{-1}(2) = \{a, c\}$.
(d) $f^{-1}(1) = \emptyset$.
(e) $f(\{a, c\}) = \{f(a), f(c)\} = \{2\}$.

When two functions are equal

Definition

Two functions f and g are **equal** *if and only if:*

- *they have the same domain D;*
- *for any $x \in D$, $f(x) = g(x)$.*

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Remark

By definition, equal functions may have different co-domains. However, they must have the same range.

Example: equal functions

Example

Are the following pairs of functions f and g equal?

(a) $f : \mathbb{R} \to \mathbb{R}$ *with* $f(x) = x$ *for all* $x \in \mathbb{R}$; $g : \mathbb{R} \to \mathbb{R}$ *with* $g(x) = \sqrt{x^2}$ *for all* $x \in \mathbb{R}$.

(b) $f : \mathbb{R} \to \mathbb{R}$ *with* $f(x) = |x|$ *for all* $x \in \mathbb{R}$; $g : \mathbb{R} \to \mathbb{R}$ *with* $g(x) = \sqrt{x^2}$ *for all* $x \in \mathbb{R}$.

(c) $f : \mathbb{Z} \to \mathbb{Z}$ *with* $f(n) = n$ *mod* 2 *for all* $n \in \mathbb{Z}$; $g : \mathbb{Z} \to \{0, 1\}$ *with*

$$g(n) = \begin{cases} 0, & \text{if } n \text{ is even;} \\ 1, & \text{if } n \text{ is odd.} \end{cases}$$

for all $n \in \mathbb{Z}$.

Example: equal functions

Solution

(a) $f(-1) = -1$ *while* $g(-1) = \sqrt{(-1)^2} = \sqrt{1} = 1$, *so* $f \neq g$.

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(b) Note that for all $x \in \mathbb{R}$, *we have that* $\sqrt{x^2} = |x|$, *so* $f = g$.

Example: equal functions

Solution

(a) $f(-1) = -1$ *while* $g(-1) = \sqrt{(-1)^2} = \sqrt{1} = 1$, *so* $f \neq g$.
(b) Note that for all $x \in \mathbb{R}$, *we have that* $\sqrt{x^2} = |x|$, *so* $f = g$.
(c) f *and* g *have the same domain and the same value for every element in the domain* $\mathbb{Z}$, *so* $f = g$.

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Consider the following relation F between $\mathbb{Q}$ and $\mathbb{Z}$ such that for all $\frac{m}{n} \in \mathbb{Q}$ with $m, n \in \mathbb{Z}$, we let $\left(\frac{m}{n}, m\right) \in F$. Is F a function?

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The answer is "no". For the element "one half" in $\mathbb{Q}$, it could be expressed as $\frac{1}{2}$. By definition $\left(\frac{1}{2}, 1\right) \in F$. in addition, one half is also equal to $\frac{2}{4}$, so $\left(\frac{1}{2}, 2\right) \in F$. But any element in the domain of a function cannot be mapped to more than one element in the co-domain, so F is not a function.

Example: well-defined function

Example

Let $F : \mathbb{Q}^+ \to \mathbb{Z}$ *such that for any* $\frac{m}{n} \in \mathbb{Q}^+$ *with* $m, n \in \mathbb{N}$, *we have that* $F\left(\frac{m}{n}\right) = m$ *div* n. *Is this* F *a well-defined function?*

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Let $F : \mathbb{Q}^+ \to \mathbb{Z}$ such that for any $\frac{m}{n} \in \mathbb{Q}^+$ with $m, n \in \mathbb{N}$, we have that $F\left(\frac{m}{n}\right) = m$ div n. Is this F a well-defined function?

Solution

Note that $m = n \cdot (m \text{ div } n) + m \text{ mod } n$. Since $0 \le m \text{ mod } n < n$, we have that

$$n \cdot (m \text{ div } n) \le m < n \cdot (m \text{ div } n + 1).$$

Hence

$$m \text{ div } n \le m/n < (m \text{ div } n) + 1.$$

It turns out that m div n is always the largest integer not exceeding $\frac{m}{n}$, so for different choices of m, n for the same positive rational number, we get the same value of F. So F is a well-defined function.

Sequences

Sequences could be viewed as functions defined on a subset of $\mathbb{Z}$: it is a function that maps n to a_n.

Power set

Recall that for any set A, $\mathscr{P}(A)$ is the power set of A. The following function F has domain $\mathscr{P}(A)$: for each $X \in \mathscr{P}(A)$, $F(X) = |X|$, the cardinality of X.
Suppose $|A| = n \in \mathbb{N}$, what would be the range of F?

Power set

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Suppose $|A| = n \in \mathbb{N}$, what would be the range of F?

Solution

The cardinality of $X \in \mathscr{P}(A)$ is at least 0 and at most n. In addition, for any integer i between 0 and n, there exists some subset of A whose cardinality is i. Hence the range is

$$\{0, 1, \dots, n\}.$$

Logarithms

Definition

Let b be a positive real number with $b \neq 1$. For each positive real number x, the **logarithm** *with base b of x, written $\log_b x$, is the exponent to which b must be raised to obtain x. Symbolically,*

$$\log_b x = y \Leftrightarrow b^y = x.$$

The **logarithmic function** *with base b is the function from R^+ to R that takes each positive real number x to $\log_b x$.*

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Remark

Why do we require $b \neq 1$? Because the power of 1 is always 1, so it cannot be a general x.

Example: logarithms

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Find the following values:

(a) $\log_3 9$;
(b) $\log_2 \frac{1}{2}$;
(c) $2^{\log_2 100}$.

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Solution

(a) Since $3^2 = 9$, $\log_3 9 = 2$.
(b) Since $2^1 = 2$, *we have that* $2^{-1} = \frac{1}{2}$. *So* $\log_2 \frac{1}{2} = -1$.

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Solution

(a) Since $3^2 = 9$, $\log_3 9 = 2$.
(b) Since $2^1 = 2$, *we have that* $2^{-1} = \frac{1}{2}$. *So* $\log_2 \frac{1}{2} = -1$.
(c) By definition, $\log_2 100$ *is a real number* y *such that* $2^y = 100$.
Hence $2^{\log_2 100} = 100$.

HW# 9 of this section

Section 7.1 Exercise 2, 4(c), 7(b)(d), 10(d)(e), 25(b), 28, 42.