Math 325K - Lecture 18
Section 7.2

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Outline

- One-to-one functions.
- Onto functions.
- Inverse functions.

Motivation

Last time we explained that for well-defined functions, different elements in the domain can be mapped to the same element in the co-domain. Sometimes we don't like this phenomenon, and we introduce a property of functions without this phenomenon.

Definition

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Let F be a function from a set X to a set Y. F is **one-to-one** *(or* **injective***) if and only if for all elements x_1 and x_2 in X, if $F(x_1) = F(x_2)$, then $x_1 = x_2$. Symbolically,*

$F : X \to Y$ *is one-to-one* $\Leftrightarrow \forall x_1, x_2 \in X, F(x_1) = F(x_2) \to x_1 = x_2$.

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$$F : X \to Y \text{ is one-to-one} \Leftrightarrow \forall x_1, x_2 \in X, F(x_1) = F(x_2) \to x_1 = x_2.$$

Remark

The one-to-one property is equivalent to "different elements in the domain have different images".

How to check one-to-one property

Given a function F, we would like to check whether it is one-to-one or not. Note that the one-to-one property is a universal statement, therefore we have:

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Given a function F, we would like to check whether it is one-to-one or not. Note that the one-to-one property is a universal statement, therefore we have:

- To justify that F is one-to-one, we need to show that for two arbitrary different elements in the domain of F, F have different values at them.
- To show that F is not one-to-one, it is enough to present a counterexample - two different elements in the domain of F that have the same value under F.

Example: one-to-one property

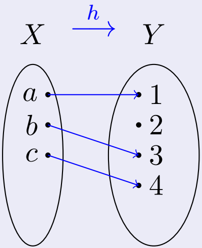
Example

Find out whether the following functions are one-to-one or not.

(a) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 4x - 1$ *for all* $x \in \mathbb{R}$.

(b) $g : \mathbb{Z} \to \mathbb{Z}$, $g(n) = n^2$ *for all* $n \in \mathbb{Z}$.

(c) h *defined by the following arrow diagram:*



Example: one-to-one property

Solution

(a) For any two elements $x_1, x_2 \in \mathbb{R}$*, we have*

$$f(x_1) = 4x_1 - 1, f(x_2) = 4x_2 - 1.$$

Suppose $f(x_1) = f(x_2)$*, then* $4x_1 - 1 = 4x_2 - 1$*, and thus* $4x_1 = 4x_2, x_1 = x_2$*. Hence* f *is one-to-one.*

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(b) Note that $g(-1) = g(1) = 1$, *so* g *is not one-to-one.*

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(b) Note that $g(-1) = g(1) = 1$, *so* g *is not one-to-one.*
(c) Note that all arrows have different ends, so different elements in X *are mapped to different elements in* Y, *thus* h *is one-to-one.*

Motivation

Recall that the range of a function is always a subset of its co-domain. As I explained last time, the elements in the set "co-domain minus range" are not involved in the function at all, so we would like to know how big is this difference set.

Definition

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Let F be a function from a set X to a set Y. F is **onto** *(or* **surjective***) if and only if given any element $y \in Y$, there exists an element $x \in X$ such that $F(x) = y$. Symbolically:*

$$F : X \to Y \text{ is onto} \Leftrightarrow \forall y \in Y, \exists x \in X \text{ such that } F(x) = y.$$

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Remark

A function is onto if and only if its range equals to its co-domain.

Example: onto property

Example

Find out whether the following functions are onto or not.

(a) $f : \mathbb{Z} \to \mathbb{Z}$ *with* $f(n) = 2n + 1$ *for all* $n \in \mathbb{Z}$.

(b) $g : \mathbb{Q}^+ \to \mathbb{Q}^+$ *with* $g(x) = \frac{1}{x}$ *for all* $x \in \mathbb{Q}^+$.

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Solution

(a) Note that when $n \in \mathbb{Z}$, $2n + 1$ *is always odd, so all even integers are not in the range of* f *and* f *is not onto.*

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Solution

(a) Note that when $n \in \mathbb{Z}$, $2n + 1$ *is always odd, so all even integers are not in the range of* f *and* f *is not onto.*
(b) For any positive rational number t, $\frac{1}{t}$ *is still a positive rational number. And note that*

$$g\left(\frac{1}{t}\right) = 1/\frac{1}{t} = t.$$

So t *belongs to the range of* g, *and thus* g *is onto.*

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- $F(n) = n$ *for all* $n \in \mathbb{N}$ *- both one-to-one and onto;*

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- $F(n) = 2n$ *for all* $n \in \mathbb{N}$ *- one-to-one but not onto;*

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- $F(n) = \lfloor \frac{n+1}{2} \rfloor$ *- onto but not one-to-one;*
- $F(n) = 1$ *for all* $n \in \mathbb{N}$ *- neither one-to-one nor onto.*

One-to-one correspondence

Definition

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Remark

Suppose $F : X \to Y$ *is a bijection and both* X *and* Y *are finite sets. Then* $|X| = |Y|$*.*

Motivation

Note that the operation by functions are always *directed* - from the domain to the co-domain. Is it possible to reverse the arrows? More specific, can we also define another function that goes from the co-domain of F to the domain of F?

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The function F should satisfy some properties:

- *for every element in the co-domain of F, it must have at least one preimage - F must be onto;*

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Remark

The function F should satisfy some properties:

- *for every element in the co-domain of F, it must have at least one preimage - F must be onto;*
- *for every element in the co-domain of F, its preimage cannot have more than 1 element - F must be one-to-one.*

As a result, we can only define inverse functions to bijective functions.

Definition

Definition

Let $F : X \to Y$ be a bijection. The **inverse function** *of F, denoted F^{-1}, is a function from Y to X with the following property: for each $y \in Y$, since F is a bijection, there is a unique element $x \in X$ such that $F(x) = y$, and we let $F^{-1}(y) = x$.*

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Remark

Unfortunately we have the abuse of notation F^{-1}. To distinguish, for any $y \in Y$, note that if it is the inverse image, then $F^{-1}(y)$ is a **subset** *of X; if it is the inverse function, then $F^{-1}(y)$ is an* **element** *of X.*

An inverse function is also a bijection

Theorem

If $F : X \to Y$ is a bijection, so is F^{-1}.

An inverse function is also a bijection

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Proof.

For any $x \in X$, by definition of inverse function we have $F^{-1}(F(x)) = x$ so F^{-1} is onto. For elements $y_1, y_2 \in Y$, suppose $F^{-1}(y_1) = F^{-1}(y_2)$. For convenience we denote this value by x. Then by definition of inverse function, $F(x) = y_1$ and $F(x) = y_2$, so $y_1 = y_2$ and F^{-1} is one-to-one. □

Example: find inverse function

Example

Find the inverse functions of the following bijections:

(a) $f : \mathbb{R} \to \mathbb{R}$ *with* $f(x) = 4x + 1$ *for all* $x \in \mathbb{R}$.

(b) $g : \mathbb{R}^+ \to \mathbb{R}$ *with* $g(x) = \log_2 x$.

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Solution

(a) For each $y \in \mathbb{R}$, f^{-1} *maps* y *to the unique real number* x *such that* $f(x) = y$. *The key step is to express* x *in terms of* y.

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Solution

(a) For each $y \in \mathbb{R}$, f^{-1} *maps* y *to the unique real number* x *such that* $f(x) = y$. *The key step is to express* x *in terms of* y. *Note that* $4x + 1 = y$, *so* $x = \frac{y-1}{4}$. *Then the inverse function of* f *is*

$$f^{-1} : \mathbb{R} \to \mathbb{R}, f^{-1}(y) = \frac{y-1}{4} \quad \forall y \in \mathbb{R}.$$

Example: find inverse function

Solution

(b) For each $y \in \mathbb{R}$, $g^{-1}(y)$ *is the unique positive real number* x *such that* $g(x) = y$. *Thus* $\log_2 x = y, 2^y = x$. *Then the inverse function of* g *is*

$$g^{-1} : \mathbb{R} \to \mathbb{R}^+, g^{-1}(y) = 2^y \quad \forall y \in \mathbb{R}.$$

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Solution

(b) For each $y \in \mathbb{R}$, $g^{-1}(y)$ *is the unique positive real number* x *such that* $g(x) = y$. *Thus* $\log_2 x = y, 2^y = x$. *Then the inverse function of* g *is*

$$g^{-1} : \mathbb{R} \to \mathbb{R}^+, g^{-1}(y) = 2^y \quad \forall y \in \mathbb{R}.$$

Remark

For bijections, the domain and the co-domain are not necessarily equal.

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Solution

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Remark

For bijections, the domain and the co-domain are not necessarily equal.

Remark

In general, the inverse functions of logarithmic functions are exponential functions.

Properties of logarithm

Since logarithmic functions are the inverse functions of exponential functions, we have the following properties.

Theorem

For any positive real numbers b, c, x, y *and with* $b, c \neq 1$ *and for all real numbers* a*:*

- (a) $\log_b xy = \log_b x + \log_b y$.
- (b) $\log_b \left(\frac{x}{y}\right) = \log_b x - \log_b y$.
- (c) $\log_b (x^a) = a \log_b x$.
- (d) $\log_c x = \frac{\log_b x}{\log_b c}$.

Example: logarithmic functions

Example

Compute

(a) $\log_{10} 5 + \log_{10} 20$;

(b) $\dfrac{\log_3 8}{\log_3 2}$.

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(a) $\log_{10} 5 + \log_{10} 20$;

(b) $\frac{\log_3 8}{\log_3 2}$.

Solution

(a) $\log_{10} 5 + \log_{10} 20 = \log_{10} (5 \cdot 20) = \log_{10} 100 = 2.$

Example: logarithmic functions

Example

Compute

(a) $\log_{10} 5 + \log_{10} 20$;

(b) $\frac{\log_3 8}{\log_3 2}$.

Solution

(a) $\log_{10} 5 + \log_{10} 20 = \log_{10}(5 \cdot 20) = \log_{10} 100 = 2$. *(b)* $\frac{\log_3 8}{\log_3 2} = \log_2 8 = 3$.

HW # 9 of this section

Section 7.2 Exercise 4, 7(b), 9(c)(d), 12(a), 25(b).

Midterm # 2 Review Outline

Sequences

- Defined as functions.
- Explicit formula of terms.
- $\sum$ and $\prod$ notations.
- Arithmetic and geometric sequences.

Mathematical induction

- The pattern: basis step & inductive step, using inductive hypothesis.
- Strong mathematical induction and the difference from the ordinary version.
- Common mistakes to avoid.
- Equivalent to the well-ordering principle for the integers.

Recursively defined functions

- Factorials.
- Fibonacci sequences.
- The function "n choose r" $\binom{n}{r}$.

Set theory

- Sets are determined by their elements.
- Operations: union, intersection, difference and complement.
- Relationships: subset, proper subset, disjoint.
- Properties and identities (De Morgan's Laws and so on).

Functions

- Defined as relations.
- Notions: domain, co-domain, range, image, preimage, inverse image.
- Properties: well-defined, one-to-one, onto, bijective, inverse functions.