Math 325K - Lecture 2
Section 2.1 Logical form and logical equivalence

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Outline

- Logical statements.
- Compound statements.
- Truth values and truth tables.
- Logical equivalence.

The logical form of arguments

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Consider the argument:
If Jane is a math major or Jane is a computer science major, then Jane will take Math 325K.
It is of the format: if p or q, then r.

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`If Jane is a math major or Jane is a computer science major, then Jane will take Math 325K.`

It is of the format: if p or q, then r.

Remark

For logic, we focus on the patterns of reasoning.

Statements

Definition

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Remark

If one cannot tell whether the sentence is true or false, or its truthfulness depends on the value of some variable, then it is not a statement.

Example of logical statements and non statements

Example

Are the following sentences statements?

(a) *The square of* 3 *is* 9.

(b) $a + b > 0$.

(c) *This M325K class has* 2 *midterm exams and it has* 1 *final exam.*

(d) *There are* 5 *named oceans in total on earth.*

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(d) *There are* 5 *named oceans in total on earth.*

Solution

(a), (c), (d) are statements where (a) and (c) are true, (d) is false. (b) is not a statement, because its truthfulness depends on the values of variables a *and* b.

Negation

Now we introduce 3 operators of statements that are used to build more complicated logical statements.

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Remark

Consistent with our common sense, when p is true, $\sim p$ must be false; and when p is false, $\sim p$ must be true.

Conjunction

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Examples of compound statements

Example

Let h be 'John is healthy', w be 'John is wealthy', and s be 'John is smart'. Rewrite the following compound statements using h, w, s and logical operators.

- (a) *John is smart but not wealthy.*
- (b) *John is healthy and wealthy but not smart.*
- (c) *John is not wealthy but he is healthy and smart.*
- (d) *John is wealthy, but he is not both healthy and smart.*

Translation to and from English

In English sentences, there are other words related to the logical operations of statements.

Remark

The word 'but' serves as 'and' in general. For example, consider the compound statement 'the building is tall but it has no elevator'. If p denotes 'the building is tall' and q denotes 'the building has no elevator', the statement is rephrased as $p \wedge q$.

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Remark

Another common phrase is 'neither ... nor ...'. Given statements p and q, the statement 'neither p nor q' is literally $(\sim p) \wedge (\sim q)$.

Examples of compound statements

Solution

(a) $s \wedge {\sim}w$.

(b) $h \wedge w \wedge {\sim}s$.

(c) ${\sim}w \wedge h \wedge s$.

(d) $w \wedge {\sim}(h \wedge s)$.

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(c) ${\sim}w \wedge h \wedge s$.

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Remark

When multiple operator appear, the priority order is $\sim\; > \wedge = \vee$.
We need to add parentheses when ambiguity exists.

Other operators

By definition, ~ is a unary operator and both $\wedge$ and $\vee$ are binary ones. There are several other logical operators that could be defined from the previous three, for example, **conditional** $\to$ and **equivalence** $\leftrightarrow$. We will discuss the conditional operator tomorrow.

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Remark

These operators are also called logical connectives because they can connect statements to compound statements.

Statement forms

Definition

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Remark

A statement form is simply a pattern that represents a lot of statements with different components.

Definition and truth tables

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Remark

These non-compound statements appeared a compound statement are called **statement variables**.

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p	$\sim p$
T	F
F	T

Figure: The truth table of negation ~.

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T	F	F
F	T	F
F	F	F

Figure: The truth table of conjunction $\wedge$.

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p	q	$p \vee q$
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T	F	T
F	T	T
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Figure: The truth table of disjunction $\vee$.

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p	q	${\sim}q$	$p \wedge {\sim}q$
T	T	F	F
T	F	T	T
F	T	F	F
F	F	T	F

Figure: *The truth table of* $p \wedge {\sim}q$.

Truth tables for more general statements

If a statement form is more complicated, for example

$$(p \vee {\sim}q) \wedge ({\sim}p \vee r),$$

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In this case, we can write the following table

p	q	r	${\sim}p$	${\sim}q$	$p \vee {\sim}q$	${\sim}p \vee r$	$(p \vee {\sim}q) \wedge ({\sim}p \vee r)$
T	F	T	F	T	T	T	T
				$\cdots$			

Figure: The truth table of $(p \vee {\sim}q) \wedge ({\sim}p \vee r)$.

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Two statements form p *and* q *are* **logically equivalent**, *denoted by* $p \equiv q$, *if and only if the following conditions are satisfied:*

1. p *and* q *have the same set of statement variables;*
2. *for every combination of the truth value of these statement variables,* p *and* q *have the same truth value.*

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Equivalently, p and q have exactly the same truth table.

Remark

One method to check whether p and q are logically equivalent is to write their truth tables.

Checking logical equivalences by truth tables

Example

Show that statement forms $p \wedge {\sim}q$ *and* ${\sim}({\sim}p \vee q)$ *are logically equivalent.*

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Show that statement forms $p \wedge {\sim}q$ *and* ${\sim}({\sim}p \vee q)$ *are logically equivalent.*

Proof.

Their truth tables are identical:

p	q	${\sim}q$	$p \wedge {\sim}q$
T	T	F	F
T	F	T	T
F	T	F	F
F	F	T	F

p	q	${\sim}p$	${\sim}p \vee q$	${\sim}({\sim}p \vee q)$
T	T	F	T	F
T	F	F	F	T
F	T	T	T	F
F	F	T	T	F

$\square$

Examples about logical equivalences

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Example

The following pairs of statement forms are logically equivalent:

1. $\sim(\sim p)$ *and* p.
2. $p \wedge q$ *and* $q \wedge p$.
3. $p \vee q$ *and* $q \vee p$.
4. $p \wedge (q \wedge r)$ *and* $(p \wedge q) \wedge r$.
5. $p \vee (q \vee r)$ *and* $(p \vee q) \vee r$.

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5. $p \vee (q \vee r)$ *and* $(p \vee q) \vee r$.

Remark

Both $\wedge$ *and* $\vee$ *satisfy the commutative and the associative laws.*

De Morgan's Laws

Augustus De Morgan was a British mathematician and logician.

Definition

De Morgan's Laws *consist of the following two pairs of logically equivalent statement forms:*

- $\sim(p \wedge q) \equiv (\sim p) \vee (\sim q)$;
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- $\sim(p \vee q) \equiv (\sim p) \wedge (\sim q)$.

Remark

We can easily verify De Morgan's Laws using truth tables.

A cautionary example

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Consider the following compound statement a: 'John is tall and thin'. What is its negation in English sentences?

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Solution

Let p be 'John is tall' and q be 'John is thin'. Then a is simply $p \wedge q$. By De Morgan's Laws, $\sim a$ is logically equivalent to $(\sim p) \vee (\sim q)$, which is 'John is not tall or not thin'. However, if we take the negation of the original sentence, it becomes 'John is not tall and thin'. How to understand this phenomenon?
The negation of a is 'John is not (tall and thin)'. So actually this $\sim a$ is not 'evaluated' at all. And if we do want to break it down, we apply De Morgan's Laws and get the correct result.

Tautologies and contradictions

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Remark

Other than truth tables, we can show a statement form is tautology or contradiction by simplifying it with logically equivalent stetment forms.

Examples of tautologies and contradictions

Example

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Show that $p \vee {\sim}p$ is a tautology and $p \wedge {\sim}p$ is a contradiction.

Proof.

Simply by truth tables.

p	${\sim}p$	$p \vee {\sim}p$
T	F	T
F	T	T

p	${\sim}p$	$p \wedge {\sim}p$
T	F	F
F	T	F

$\square$