Math 325K - Lecture 20
Section 7.4 Cardinality and size of infinity

Bo Lin

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Outline

- Cardinality of sets.
- Countable size $\aleph_0$.
- Uncountable size $\aleph_1$ and more.

Motivation

Exercise

Let $X = \{1, 2, 3, 4, 5\}$ *and* $Y = \{a, b, c, d, e\}$. *Do the two sets have equal number of elements?*

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Solution

They do, as they both have 5 elements.

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Solution

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Remark

There is an alternative way to check: one can map 1 to a, 2 to b, and so on. In fact there exists a bijection between elements in X and Y.

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Proposition

For finite sets A and B, they have the same cardinality if and only if they have the same number of elements.

"Same cardinality" is an equivalence relation

Theorem

For all sets A, B and C, we have the following properties:

- *(reflexive) A and A have the same cardinality;*
- *(symmetric) if A and B have the same cardinality, then B and A have the same cardinality;*
- *(transitive) if A and B have the same cardinality and B and C have the same cardinality, then A and C have the same cardinality.*

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Reflexive: I_A works. Symmetric: let $f : A \to B$ be a one-to-one correspondence, then f^{-1} is also a one-to-one correspondence from B to A. Transitive: let $f : A \to B$ and $g : B \to C$ be the one-to-one correspondences, then $g \circ f$ is also one-to-one and onto, so it is a one-to-one correspondences from A and C. □

Infinite sets and finite sets do not have the same cardinality

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Remark

Then we would like to study the cardinality of infinite sets. A natural question would be: do all infinite sets have the same cardinality?

A counterintuitive property of infinite sets

Example

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Proposition

$\mathbb{N}$ *and* $2\mathbb{N}$ *have the same cardinality.*

Corollary

An infinite set and a proper subset of it can have the same cardinality.

Countable sets

Definition

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($\aleph$ *is the first letter in Hebrew alphabet)*

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Note that we can count all elements of $\mathbb{N}$ one by one, we have the following definition.

Definition

If a set and $\mathbb{N}$ have the same cardinality, then it is **countably infinite**. *A set is* **countable** *if it is either finite or countably infinite.*

Then an important question is: what infinite sets are countable?

Exercise: $\mathbb{Z}$ is countable

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Proof.

Suppose we can write all integers in a sequence such that every integers appears exactly once in the sequence, then we are done. Because we can let $f : \mathbb{N} \to \mathbb{Z}$ be a function such that $f(n)$ is the n-th term in the sequence. One example of the sequence is:

$$0, 1, -1, 2, -2, 3, -3, \cdots$$

□

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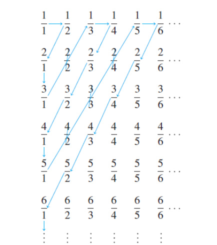

Properties of countable sets

Theorem

The following types of sets are countable:

1. *the subset of a countable set;*
2. *the union of a countable set and a finite set;*
3. *the union of finitely many countable sets;*
4. *the union of countably many countable sets, which means it is the union of an infinite family of sets $S_1, S_2, \cdots$ such that each S_i is a countable set.*

Properties of countable sets

Proof.

(i) Let A be a countable set and $B \subset A$. If B is finite, we are done. If B is infinite, so is A and we can list all elements in A as $a_1, a_2, \dots, a_n, \dots$. Now we need show that B is countably infinite. We recursively define a function $g : \mathbb{N} \to B$. Let $S_1 = \{n \in \mathbb{N} \mid a_n \in B\}$. Since B is nonempty, so is S_1. By the well-ordering principle for the integers, S_1 contains a least element i_1, and we let $g(1) = a_{i_1}$. For $k \geq 2$, suppose $g(k-1)$ is already defined, let $S_k = \{n \in \mathbb{N} \mid n > i_{k-1}, a_n \in B\}$. Since B is nonempty, so is S_k. Once again S_k has a least element i_k and we let $g(k) = a_{i_k}$. Then this function g is a bijection between $\mathbb{N}$ and B. □

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(ii)(iii) Note that they are both subsets of the union of countably many countable sets, they follow from (iv) and (i).
(iv) This is very similar to the proof of $\mathbb{Q}^+$ being countable. Let $S_1, S_2, \cdots$ be countably many sets, each one is countable. Then we may assume that $S_n = \{a_{n,1}, a_{n,2}, a_{n,3}, \cdots\}$ for each $n \in \mathbb{N}$. We can list them in s sequence like

$$a_{1,1}, a_{1,2}, a_{2,1}, a_{1,3}, a_{2,2}, a_{3,1}, \ldots$$

Then delete redundant elements if necessary, we prove that

$$\bigcup_{i=1} S_i$$

is countable too. □

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Corollary

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Cantor's diagonalization process

Proof.

Suppose we can list all real numbers between 0 and 1 in a sequence $r_1, r_2, r_3, \ldots$. Let the decimal presentation of r_i be

$$0.a_{i1}a_{i2}a_{i3}\ldots$$

Here each a_{ij} is an integer between 0 and 9.

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such that each b_i is different from a_{ii} (there are 10 possible choices of the digit, so this is always doable).

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Proposition

For any real numbers $a < b$*, the interval* $(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$ *has cardinality* $\aleph_1$*.*

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Proposition

For any real numbers $a < b$, the interval $(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$ has cardinality $\aleph_1$.

Proof.

The trigonometric function $\tan(x)$ gives a bijection between $(-\frac{\pi}{2}, \frac{\pi}{2})$ and $\mathbb{R}$. And a bijection between (a, b) and $(-\frac{\pi}{2}, \frac{\pi}{2})$ could be easily established by a linear function. □

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Theorem

Suppose the cardinality of a set S is $\aleph$, then $\mathcal{P}(S)$ has a greater cardinality (usually denoted $2^{\aleph}$).

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Suppose the cardinality of a set S is $\aleph$, then $\mathcal{P}(S)$ has a greater cardinality (usually denoted $2^{\aleph}$).

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There is no largest cardinality of sets.

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Proof.

It suffices to show that there is no bijection between S and $\mathcal{P}(S)$. Suppose a function $\phi : S \to \mathcal{P}(S)$ is a bijection. We consider the following subset of S:

$$T = \{x \in S \mid x \notin \phi(x)\}.$$

Since $T \in \mathcal{P}(S)$, there exists $y \in S$ such that $T = \phi(y)$. Now we check whether $y \in T$.

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Since $T \in \mathcal{P}(S)$, there exists $y \in S$ such that $T = \phi(y)$. Now we check whether $y \in T$. If $y \in T$, by the definition of T, y is an element of S such that $y \notin \phi(y) = T$, a contradiction! Conversely, if $y \notin T$, by the definition of T, y is an element of S such that $y \notin \phi(y)$ does not hold, so $y \in \phi(y) = T$, still a contradiction! □

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There is no other cardinality between $\aleph_0$ and $\aleph_1$.

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Remark

After hard work by several generations of scholars, it turned out that under our system of axioms, the continuum hypothesis can neither be proved nor disproved, so we can add it or its negation as a new axiom.

HW # 10 of this section

Exercise 5, 15, 17.