Math 325K - Lecture 21
Section 8.1 & 8.2

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November 15th, 2018

Outline

- Relations and their inverses.
- Directed graph of relations.
- Properties of relations.

Definition

Recall the definition of relations (copied from Lecture 1)

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Let A and B be sets. A **relation** *R from A to B is a subset of $A \times B$. Given an ordered pair (x, y) in $A \times B$, x is related to y by R, written $x \, R \, y$, if and only if (x, y) is in R.*

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Remark

When we study the properties of a set, we may need to consider relations on it.

Inverse of relations

Just like we defined inverse function of bijections, we have an analogue for relations.

Definition

Let R be a relation from A to B. Define the **inverse relation** R^{-1} *from B to A as follows:*

$$R^{-1} = \{(y, x) \in B \times A \mid (x, y) \in R\}.$$

In other words, R^{-1} is a relation such that $y\, R^{-1}\, x$ if and only if $x\, R\, y$.

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Remark

Note that every relation has a unique inverse, which is different from the case of functions!

Exercise: inverse of "divides"

Exercise

Let $A = \{2, 3, 4\}$ *and* $B = \{5, 6, 7, 8\}$ *and let* R *be the "divides" relation from* A *to* B*: For all* $(x, y) \in A \times B$,

$$x \, R \, y \Leftrightarrow x \mid y.$$

Find the ordered pairs in R^{-1}*, and describe* R^{-1} *in words.*

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Solution

First, we find all ordered pairs in R*, which are* $(2, 6), (2, 8), (3, 6), (4, 8)$. *Then the ordered pairs in* R^{-1} *are*

$$(6, 2), (8, 2), (6, 3), (8, 4).$$

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Note that $y \, R^{-1} \, x$ *if and only if* x *divides* y*, then how to describe this relation from* y *to* x*?*

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Note that $y \, R^{-1} \, x$ *if and only if* x *divides* y*, then how to describe this relation from* y *to* x*? We simply say that* y *is a multiple of* x*.*

Arrow diagram for relations

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Definition

A **directed graph** *of a relation R on a set A is the following figure: we encompass points corresponding to elements of A by an ellipse or a circle, and for every pair $(x, y) \in R$, we draw an arrow from x to y.*

Example: directed graph

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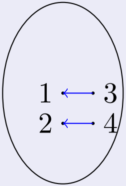

Exercise: direct graph of "equal residue"

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Let $A = \{3, 4, 5, 6, 7, 8\}$ *and relation* R *on* A *be as follows: For all* $x, y \in A$,

$$x \, R \, y \Leftrightarrow 3 \mid (x - y) \,.$$

Draw the directed graph of R.

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$$x \, R \, y \Leftrightarrow 3 \mid (x - y) \, .$$

Draw the directed graph of R.

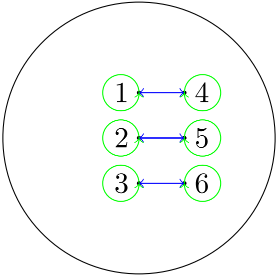
Solution

The ordered pairs in R *are*

$$(3,3), (3,6), (4,4), (4,7), (5,5), (5,8),$$
$$(6,3), (6,6), (7,4), (7,7), (8,5), (8,8).$$

Exercise: direct graph of "equal residue"

So the directed graph is the following figure:



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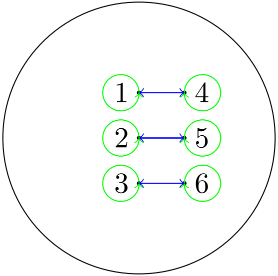

Remark

Don't forget the loops $x \to x$*!*

The properties of equivalence relation

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Definition

Let R be a relation on a set A. R is called

- **reflexive**, *if for all $x \in A$, $x \, R \, x$;*
- **symmetric**, *if for all $x, y \in A$, $x \, R \, y$ implies $y \, R \, x$;*
- **transitive**, *if for all $x, y, z \in A$, the conjunction of $x \, R \, y$ and $y \, R \, z$ implies $x \, R \, z$.*

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- **transitive**, *if for all $x, y, z \in A$, the conjunction of $x \, R \, y$ and $y \, R \, z$ implies $x \, R \, z$.*

Definition

If a relation R satisfies all these 3 properties, then it is called an **equivalence relation** *on A.*

Examples of equivalence relations

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On any subset of $\mathbb{Z}$ and for every $n \in \mathbb{N}$, the relation $x\,R\,y$ if and only if $n \mid (x - y)$ is an equivalence relation.

On any set of sets, "having the same cardinality" is an equivalence relation.

Exercise: examples of relations

Exercise

Consider the following relations R defined on $\mathbb{N}$, are they reflexive, symmetric and transitive?

- (a) *For all $x, y \in \mathbb{N}$, $x \, R \, y$ if and only if $x \mid y$.*
- (b) *For all $x, y \in \mathbb{N}$, $x \, R \, y$ if and only if $x < y$.*
- (c) *For all $x, y \in \mathbb{N}$, $x \, R \, y$ if and only if $x + y$ is even.*

Exercise: examples of relations

Solution

Relation	*Reflexive*	*Symmetric*	*Transitive*
(a)	*Yes*	*No* $(x = 1, y = 2)$	*Yes*

Exercise: examples of relations

Solution

Relation	*Reflexive*	*Symmetric*	*Transitive*
(a)	*Yes*	*No* $(x = 1, y = 2)$	*Yes*
(b)	*No* $(x = 1)$	*No* $(x = 1, y = 2)$	*Yes*

Exercise: examples of relations

Solution

Relation	*Reflexive*	*Symmetric*	*Transitive*
(a)	*Yes*	*No* $(x = 1, y = 2)$	*Yes*
(b)	*No* $(x = 1)$	*No* $(x = 1, y = 2)$	*Yes*
(c)	*Yes*	*Yes*	*Yes*

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Find a relation R on $\mathbb{N}$ that is reflexive and symmetric, but not transitive.

Solution

We let R be a relation on $\mathbb{N}$ such that for all $x, y \in \mathbb{N}$, $x \, R \, y \Leftrightarrow |x - y| \leq 1$. This R is reflexive because for any $x \in \mathbb{N}$, $|x - x| = 0 \leq 1$ and if $|x - y| \leq 1$, then $|y - x| \leq 1$.

Exercise: the properties are independent of each other

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We let R be a relation on $\mathbb{N}$ such that for all $x, y \in \mathbb{N}$, $x \, R \, y \Leftrightarrow |x - y| \leq 1$. This R is reflexive because for any $x \in \mathbb{N}$, $|x - x| = 0 \leq 1$ and if $|x - y| \leq 1$, then $|y - x| \leq 1$. However we have $1 \, R \, 2, 2 \, R \, 3$, but $2 \not R \, 3$. So R is not transitive.

Properties on R^{-1}

Theorem

Let R be a relation on set A. Then we have

- (a) R *is reflexive* $\Leftrightarrow$ R^{-1} *is reflexive;*
- (b) R *is symmetric* $\Leftrightarrow$ R^{-1} *is symmetric;*
- (c) R *is transitive* $\Leftrightarrow$ R^{-1} *is transitive.*

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Theorem

Let R be a relation on set A. Then we have

- (a) *R is reflexive $\Leftrightarrow$ R^{-1} is reflexive;*
- (b) *R is symmetric $\Leftrightarrow$ R^{-1} is symmetric;*
- (c) *R is transitive $\Leftrightarrow$ R^{-1} is transitive.*

Proof.

We only prove the hardest (c). Since $\left(R^{-1}\right)^{-1} = R$, it suffices to show that if R is transitive, then R^{-1} is transitive. Suppose R is transitive. For any $x, y, z \in A$, suppose $x\, R^{-1}\, y$ and $y\, R^{-1}\, z$, then $y\, R\, x$ and $z\, R\, y$. Since R is transitive, we have $z\, R\, x$. Then by definition we have $x\, R^{-1}\, z$, hence R^{-1} is transitive too. $\square$

Exercise: operations on relations

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Let both R and S be relations on a set A. If R and S are both reflexive, is $R \cap S$ always reflexive?

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Solution

Let's explore. To check whether $R \cap S$ is always reflexive, we just need to check the following universal statement:

$$\forall x \in A,\ (x, x) \in R \cap S.$$

So we choose an arbitrary element $x \in A$, and we need to check whether $(x, x) \in R$ and $(x, x) \in S$. Since R is reflexive, by definition we have $(x, x) \in R$; since S is reflexive, we also have $(x, x) \in S$. Then $(x, x) \in R \cap S$, and the answer is yes.

HW# 10 in today's sections

Section 8.1 Exercise 9(c), 11.
Section 8.2 Exercise 5, 16, 21, 39, 40.