Math 325K - Lecture 22
Section 8.3 Equivalence relation

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Outline

- Equivalence relations induced by partitions.
- Equivalence classes and representatives.
- Examples.

Recall: definition

Definition

A relation defined on a set A is an **equivalence relation** *if it is reflexive, symmetric and transitive.*

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Remark

If a relation on A is an equivalence relation, then it divides elements in A into disjoint groups. And we want to study this phenomenon in details.

Relations induced by partitions

Definition

Given a partition of a set A, the **relation induced by the partition**, R, *is defined on A as follows: For all $x, y \in A$, $x \, R \, y \Leftrightarrow$ there is a subset A_i of the partition such that both x and y are in A_i.*

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Remark

Note that this relation R is completely determined by the partition of A.

Relations induced by partitions are equivalence relations

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Symmetric: for any $x, y \in A$, if x and y belong to same subset in the partition, then y and x belong to same subset in the partition.

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Theorem

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Proof.

Reflexive: for any $x \in A$, x and itself belong to the same subset in the partition.
Symmetric: for any $x, y \in A$, if x and y belong to same subset in the partition, then y and x belong to same subset in the partition.
Transitive: for any $x, y, z \in A$, if x and y belong to same subset in the partition, and y and z belong to same subset in the partition, then the two sets mentioned are the same. So x and z belong to same subset in the partition. □

Exercise: find the partition

Exercise

Let $A = \{1, 2, 3, 4, 5\}$ *and* R *be an equivalence relation defined on* A *such that* $x \, R \, y$ *if and only if* $2 \mid (x - y)$. *If* R *is also induced by a partition on* A, *find the partition.*

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Solution

Note that the ordered pairs in R *are*

$$(1,1),(1,3),(1,5),(2,2),(2,4),(3,1),(3,3),(3,5),$$
$$(4,2),(4,4),(5,1),(5,3),(5,5).$$

So the partition is

$$A = \{1,3,5\} \cup \{2,4\}.$$

Equivalence classes

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Definition

Suppose A is a set and R is an equivalence relation on A. For each element $a \in A$, the **equivalence class** *of a, denoted $[a]$ and called the class of a for short, is the set of all elements $x \in A$ such that x is related to a by R. In symbols:*

$$[a] = \{x \in A \mid x \, R \, a\}.$$

Exercise: related elements represent the same equivalence class

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Suppose R is an equivalence relation on a set A and $x, y \in A$ such that $x \, R \, y$. Prove that $[x] = [y]$.

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Proof.

By symmetry, it suffices to prove that $[x] \subseteq [y]$. For any $z \in [x]$, we have $z \, R \, x$. Since R is transitive and we also have $x \, R \, y$, we have $z \, R \, y$. By the definition of equivalence classes, $z \in [y]$. So $[x] \subseteq [y]$. □

Representatives of equivalence classes

Definition

Suppose R is an equivalence relation on a set A and S is an equivalence class of R. A **representative** *of the class S is any element $a \in A$ such that $[a] = S$.*

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Remark

By the previous exercise, any element in the same equivalence class serves as its representative.

Property of equivalence classes

Proposition

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Proof.

We prove by contraposition. Suppose there is an element $x \in S_1 \cap S_2$. By definition there is an element $a_1 \in A$ such that $S_1 = [a_1]$. Since $x \in S_1$, $x \, R \, a_1$. By the previous exercise $[x] = [a_1] = S_1$. For the same reason, $[x] = S_2$. Hence $S_1 = S_2$, which is a contradiction. □

Rational numbers

Example

We can define $\mathbb{Q}$ *alternatively as the set of some equivalence classes: Let* $A = \mathbb{Z} \times \mathbb{Z} - \{0\}$*. Define a relation* R *on* A *such that for all* $(a, b), (c, d) \in A$,

$$(a, b)\ R\ (c, d) \Leftrightarrow ad = bc.$$

Congruence

Definition

Let m and n be integers and let d be a positive integer. We say that m is congruent to n modulo d and write

$$m \equiv n \pmod{d}$$

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Remark

As we have shown, congruences modulo any positive integer d are equivalence relations.

Exercise: congruence classes modulo 4

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Exercise

Find all congruence classes modulo 4.

Exercise: congruence classes modulo 4

Solution

The congruence classes are characterized by the residue when divided by 4. *So there are* 4 *congruence classes modulo* 4 *which form a partition of* $\mathbb{Z}$*:*

$$\{4k \mid k \in \mathbb{Z}\};$$
$$\{4k+1 \mid k \in \mathbb{Z}\};$$
$$\{4k+2 \mid k \in \mathbb{Z}\};$$
$$\{4k+3 \mid k \in \mathbb{Z}\}.$$

Exercise: congruence classes modulo 4

Solution

The congruence classes are characterized by the residue when divided by 4. So there are 4 congruence classes modulo 4 which form a partition of $\mathbb{Z}$:

$$\{4k \mid k \in \mathbb{Z}\};$$
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$$\{4k+2 \mid k \in \mathbb{Z}\};$$
$$\{4k+3 \mid k \in \mathbb{Z}\}.$$

Remark

For any positive integer d, there are d congruence classes modulo d.

HW # 11 of this section

Exercise 2(b), 12, 14(b), 20, 33, 39.