Math 325K - Lecture 24
Section 9.1 & 9.2

Bo Lin

November 29th, 2018

Outline

- Probability.
- The multiplication rule.
- Permutations and r-permutations.

Motivation

Everyday we encounter a lot events with multiple possible outcomes, for example:

- The weather tomorrow (sunny, cloudy, rainy, snowy, etc.);
- The time it takes to wait for the next bus/elevator;
- The price of the shares you have in stock market tomorrow . . .

Motivation

Everyday we encounter a lot events with multiple possible outcomes, for example:

- The weather tomorrow (sunny, cloudy, rainy, snowy, etc.);
- The time it takes to wait for the next bus/elevator;
- The price of the shares you have in stock market tomorrow . . .

Even though the outcome is not fixed, there are some patterns behind those events, for example how likely will each outcome happen. It is very important to study these patterns. That is the task of a branch of mathematics - **probability theory**.

Motivation

Everyday we encounter a lot events with multiple possible outcomes, for example:

- The weather tomorrow (sunny, cloudy, rainy, snowy, etc.);
- The time it takes to wait for the next bus/elevator;
- The price of the shares you have in stock market tomorrow ...

Even though the outcome is not fixed, there are some patterns behind those events, for example how likely will each outcome happen. It is very important to study these patterns. That is the task of a branch of mathematics - **probability theory**.
In many cases, the first thing we need to do is to figure out how many possible outcomes are there in total? So counting the cardinality of sets is also very important.

Sample spaces and events

Definition

A **sample space** *is the set of all possible outcomes of a random process or experiment. An* **event** *is a subset of a sample space.*

Sample spaces and events

Definition

A **sample space** *is the set of all possible outcomes of a random process or experiment. An* **event** *is a subset of a sample space.*

Example

Suppose one tosses a coin and the outcome could be either head or tail. In this example, the sample space is the set $\{head, tail\}$*. Say we care about "getting a head", then it is actually an event - a subset* $\{head\}$ *of the sample space.*

Equally likely probability

Definition

For a finite set A, let $N(A)$ be its cardinality (number of elements).

Equally likely probability

Definition

For a finite set A, let $N(A)$ be its cardinality (number of elements).

Let's consider the simplest case where the sample space is finite and all outcomes are equally likely to happen.

Definition (Equally likely probability formula)

If S is a finite sample space in which all outcomes are equally likely and E is an event in S, then the **probability** *of E, denoted $P(E)$, is $\frac{N(E)}{N(S)}$.*

Equally likely probability

Definition

For a finite set A, let $N(A)$ be its cardinality (number of elements).

Let's consider the simplest case where the sample space is finite and all outcomes are equally likely to happen.

Definition (Equally likely probability formula)

If S is a finite sample space in which all outcomes are equally likely and E is an event in S, then the **probability** *of E, denoted $P(E)$, is $\frac{N(E)}{N(S)}$.*

Remark

This formula is one axiom of probability theory and many problems become a matter of counting.

Exercise: playing cards

One common example in probability theory is the playing cards. Each deck consists of 52 cards, in 4 suits - spades $\spadesuit$, hearts $\heartsuit$, diamonds $\diamondsuit$ and clubs $\clubsuit$. Each suit consists of $52/4 = 13$ cards, with numbers 2 through 10, Ace (for 1), Jack (J, for 11), Queen (Q, for 12), and King (K, for 13). Spades and clubs are in black color, and hearts and diamonds are in red colors. The J, Q, K cards are called face cards.

Exercise: playing cards

One common example in probability theory is the playing cards. Each deck consists of 52 cards, in 4 suits - spades $\spadesuit$, hearts $\heartsuit$, diamonds $\diamondsuit$ and clubs $\clubsuit$. Each suit consists of $52/4 = 13$ cards, with numbers 2 through 10, Ace (for 1), Jack (J, for 11), Queen (Q, for 12), and King (K, for 13). Spades and clubs are in black color, and hearts and diamonds are in red colors. The J, Q, K cards are called face cards.

Exercise

Suppose the sample space is just the set of the 52 playing cards. Now one randomly chooses one card from the deck with equal likelihood of the cards. What is the event that the chosen card is a black face card? What is the probability of this event?

Exercise: playing cards

Solution

The event is just the subset consisting of all black face cards, which is

$$BF = \{\spadesuit J, \spadesuit Q, \spadesuit K, \clubsuit J, \clubsuit Q, \clubsuit K\}.$$

Let PC be the sample space of the 52 cards. By the equally likely probability formula,

$$P(BF) = \frac{N(BF)}{N(PC)} = \frac{6}{52} \approx 11.5\%.$$

Exercise: playing cards

Solution

The event is just the subset consisting of all black face cards, which is

$$BF = \{\spadesuit J, \spadesuit Q, \spadesuit K, \clubsuit J, \clubsuit Q, \clubsuit K\}.$$

Let PC be the sample space of the 52 cards. By the equally likely probability formula,

$$P(BF) = \frac{N(BF)}{N(PC)} = \frac{6}{52} \approx 11.5\%.$$

Remark

In many cases, the cardinality of the sample space is either given or very easy to compute, and the key to a solution is to find the cardinality of some events.

Example: counting a process with multiple steps

Exercise

Suppose during a weekend, our football team, volleyball team and soccer team each plays a regular season game. Both football and volleyball games must have a winner, while the soccer game could be a draw. How many possible outcomes for the series of these 3 *games?*

Example: counting a process with multiple steps

Solution

Note that there are 2 possible outcomes for football and volleyball and 3 for soccer. We have the following table of all possible outcomes.

Football	W	W	W	W	W	W	L	L	L	L	L	L
Volleyball	W	W	W	L	L	L	W	W	W	L	L	L
Soccer	W	D	L	W	D	L	W	D	L	W	D	L

The answer is 12, which is $2 \cdot 2 \cdot 3$.

Example: counting a process with multiple steps

Solution

Note that there are 2 possible outcomes for football and volleyball and 3 for soccer. We have the following table of all possible outcomes.

Football	*W*	*W*	*W*	*W*	*W*	*W*	*L*	*L*	*L*	*L*	*L*	*L*
Volleyball	*W*	*W*	*W*	*L*	*L*	*L*	*W*	*W*	*W*	*L*	*L*	*L*
Soccer	*W*	*D*	*L*	*W*	*D*	*L*	*W*	*D*	*L*	*W*	*D*	*L*

The answer is 12, which is $2 \cdot 2 \cdot 3$.

Remark

Note that we get an outcome of the football game first, then independently we get another result of the volleyball game, and finally the soccer game. In general, there is a pattern for such scenarios with independent steps.

The Multiplication Rule

Definition

If an operation consists of k steps and

- *the first step can be performed in n_1 ways,*
- *the second step can be performed in n_2 ways [regardless of how the first step was performed],*
- $\cdots$
- *the k-th step can be performed in n_k ways [regardless of how the preceding steps were performed],*

then the entire operation can be performed in

$$\prod_{i=1}^{k} n_i = n_1 n_2 \cdots n_k$$

ways. This is the **multiplication rule** *of counting.*

The Multiplication Rule

Definition

If an operation consists of k steps and

- *the first step can be performed in n_1 ways,*
- *the second step can be performed in n_2 ways [regardless of how the first step was performed],*
- $\cdots$
- *the k-th step can be performed in n_k ways [regardless of how the preceding steps were performed],*

then the entire operation can be performed in

$$\prod_{i=1}^{k} n_i = n_1 n_2 \cdots n_k$$

ways. This is the **multiplication rule** *of counting.*

Exercise: two letter postal codes

Exercise

USPS uses a two-letter code for every US state (and territory). For example, the code for Texas is TX. Suppose both letters in such codes could be any letter in the English alphabet. How many possible two-letter postal codes are there in total? (Of course many of them are not actually in use)

Exercise: two letter postal codes

Exercise

USPS uses a two-letter code for every US state (and territory). For example, the code for Texas is TX. Suppose both letters in such codes could be any letter in the English alphabet. How many possible two-letter postal codes are there in total? (Of course many of them are not actually in use)

Solution

We apply the multiplication rule. In order to get a two-letter code, we need to fix the first letter and then fix the second letter.

Exercise: two letter postal codes

Exercise

USPS uses a two-letter code for every US state (and territory). For example, the code for Texas is TX. Suppose both letters in such codes could be any letter in the English alphabet. How many possible two-letter postal codes are there in total? (Of course many of them are not actually in use)

Solution

We apply the multiplication rule. In order to get a two-letter code, we need to fix the first letter and then fix the second letter. The first letter could be any of the 26 *letters, so there are* $n_1 = 26$ *ways for the first step. Similarly, there are* $n_2 = 26$ *steps for the second step. So the answer is just* $26 \cdot 26 = 676$.

Exercise: IPv4 addresses

Exercise

In the past, the address of each device in the Internet is called the IPv4 address. One example is $129.116.1.0$. *Each address consists of* 4 *parts, and each part is an integer between* 0 *and* 255. *How many possible IPv4 addresses are there?*

Exercise: IPv4 addresses

Exercise

In the past, the address of each device in the Internet is called the IPv4 address. One example is $129.116.1.0$*. Each address consists of* 4 *parts, and each part is an integer between* 0 *and* 255*. How many possible IPv4 addresses are there?*

Solution

To fix an IPv4 address, we need 4 *steps, and there are* $255 + 1 = 256 = 2^8$ *ways for each step. So the answer is*

$$256^4 = 2^{8 \cdot 4} = 2^{32} \approx 4.2 \text{ billion.}$$

Exercise: IPv4 addresses

Exercise

In the past, the address of each device in the Internet is called the IPv4 address. One example is $129.116.1.0$*. Each address consists of* 4 *parts, and each part is an integer between* 0 *and* 255*. How many possible IPv4 addresses are there?*

Solution

To fix an IPv4 address, we need 4 *steps, and there are* $255 + 1 = 256 = 2^8$ *ways for each step. So the answer is*

$$256^4 = 2^{8 \cdot 4} = 2^{32} \approx 4.2 \text{ billion.}$$

Remark

Ideally, everyone should get one address, while there are already more than 7 *billion population in the world. As a result, the mechanism of IPv4 addresses is outdated and we are superseding it with the IPv6 addresses.*

Example: positions for interns

Example

Suppose a company selects three college students for internship. There is one position in each of the following three departments: marketing, technology and management and the HR chief needs to send one student to each department. How many possible arrangements are there?

Example: positions for interns

Example

Suppose a company selects three college students for internship. There is one position in each of the following three departments: marketing, technology and management and the HR chief needs to send one student to each department. How many possible arrangements are there?

Solution

One might consider the multiplication rule. We send the first student, then the second, and finally the third. There are 3 ways for the first student, and it seems 3 ways for the second student as well ... (to be continued)

Example: positions for interns

Solution

(continued) Here is the tricky point. Since each department only takes one intern, the steps are not independent. For example, if the first student is sent to the marketing department, then the second student cannot be sent to it.

Example: positions for interns

Solution

(continued) Here is the tricky point. Since each department only takes one intern, the steps are not independent. For example, if the first student is sent to the marketing department, then the second student cannot be sent to it. However, one can still apply the multiplication rule, as regardless the position of the first student, the second student always has two possible departments to go. Similarly, there is always one choice for the third student. So the answer is $3 \cdot 2 \cdot 1 = 3! = 6$.

Example: positions for interns

Solution

(continued) Here is the tricky point. Since each department only takes one intern, the steps are not independent. For example, if the first student is sent to the marketing department, then the second student cannot be sent to it. However, one can still apply the multiplication rule, as regardless the position of the first student, the second student always has two possible departments to go. Similarly, there is always one choice for the third student. So the answer is $3 \cdot 2 \cdot 1 = 3! = 6$.

Remark

This is a very common pattern in practice and we have a name for it.

Permutations

Definition

A **permutation** *of a set of objects is an ordering of the objects in a row.*

Permutations

Definition

A **permutation** *of a set of objects is an ordering of the objects in a row.*

Proposition

For any integer $n \geq 1$, the number of permutations of a set with n elements is $n!$.

The proof is just an application of the multiplication rule.

Permutations of Selected Elements

A more general case is the permutations of selected elements.

Definition

*An r-**permutation** of a set of n elements is an ordered selection of r elements taken from the set of n elements. The number of r-permutations of a set of n elements is denoted $P(n, r)$.*

Permutations of Selected Elements

A more general case is the permutations of selected elements.

Definition

*An r-**permutation** of a set of n elements is an ordered selection of r elements taken from the set of n elements. The number of r-permutations of a set of n elements is denoted $P(n, r)$.*

Example

$P(n, 1) = n, P(n, n) = n!$ *for each positive integer* n.

Exercise: license plates

Exercise

Suppose in a state, each license plate of vehicles is a sequence of three letters followed by four digits. How many license plates are possible in which all letters and digits are distinct?

Exercise: license plates

Exercise

Suppose in a state, each license plate of vehicles is a sequence of three letters followed by four digits. How many license plates are possible in which all letters and digits are distinct?

Solution

First we apply the multiplication rule. We determine the letters first and then the digits.

Exercise: license plates

Exercise

Suppose in a state, each license plate of vehicles is a sequence of three letters followed by four digits. How many license plates are possible in which all letters and digits are distinct?

Solution

First we apply the multiplication rule. We determine the letters first and then the digits. For the letters, we select the first one, which has 26 *choices. For the second letter, it could be anything but the first letter, so it has* $26 - 1 - 25$ *choices. And the third letter has* 24 *choices. Similarly, the four digits have* $10, 9, 8, 7$ *choices in total. So the answer is*

$$P(26,3) \cdot P(10,4) = 26 \cdot 25 \cdot 24 \cdot 10 \cdot 9 \cdot 8 \cdot 7 = 78,624,000.$$

Formula of $P(n, r)$

Theorem

$$P(n,r) = n(n-1)(n-2)\cdots(n-r+1) = \frac{n!}{(n-r)!}.$$

Formula of $P(n,r)$

Theorem

$$P(n,r) = n(n-1)(n-2)\cdots(n-r+1) = \frac{n!}{(n-r)!}.$$

Proof.

We apply the multiplication rule. There are n choices for the first element, then $n-1$ choices for the second element, and so on. As for the last element, it could be anything but the previously chosen $r-1$ elements, so it has $n-(r-1)$ choices. Then we have the above formula. □

Exercise: evaluate $P(n, r)$

Exercise

Evaluate $P(5, 3)$ *and* $P(6, 3)$.

Exercise: evaluate $P(n, r)$

Exercise

Evaluate $P(5,3)$ *and* $P(6,3)$.

Solution

$P(5,3) = \frac{5!}{3!} = \frac{120}{6} = 20.$

Exercise: evaluate $P(n, r)$

Exercise

Evaluate $P(5, 3)$ *and* $P(6, 3)$.

Solution

$P(5, 3) = \frac{5!}{3!} = \frac{120}{6} = 20$. $P(6, 3) = \frac{6!}{3!} = \frac{720}{6} = 120$.

HW #12 for today's sections

Section 9.1 Exercise 4, 12(b)(ii), 19(b), 20.
Section 9.2 Exercise 5, 15, 31(c).