Math 325K - Lecture 25
Section 9.3 & 9.4

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Outline

- The addition and difference rules.
- The inclusion/exclusion rule.
- The pigeonhole principle.

Example: 6 in backgammon

Example

Backgammon is one of the most ancient boardgames which is still very popular in the Mediterranean region. In each turn, one player rolls two dice at once. So the outcome is an ordered pair in $[6] \times [6]$, *where* $[6] = \{1, 2, 3, 4, 5, 6\}$.

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Remark

This counting task is a little challenging because there are multiple ways to get 6 *steps, either direct or indirect. So we cannot apply the multiplication rule. Instead, we check them separately.*

Example: 6 in backgammon

Solution

The case of a direct 6*: this means exactly that there is at least a* 6 *in the ordered pair. If the first roll is* 6*, there are* 6 *outcomes; if the second roll is* 6*, there are* 6 *outcomes. But there is an outcome* $(6, 6)$ *being counted twice. So there are actually* $6 + 6 - 1 = 11$ *outcomes in this case.*

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The case of an indirect 6 *is more complicated. First, all such pairs do not have any* 6*. If the pair contains distinct rolls, the sum of the two numbers must be* 6*. So it must be one of the following:*

$$(1, 5), (2, 4), (4, 2), (5, 1).$$

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The case of a direct 6: this means exactly that there is at least a 6 in the ordered pair. If the first roll is 6, there are 6 outcomes; if the second roll is 6, there are 6 outcomes. But there is an outcome $(6, 6)$ being counted twice. So there are actually $6 + 6 - 1 = 11$ outcomes in this case.
The case of an indirect 6 is more complicated. First, all such pairs do not have any 6. If the pair contains distinct rolls, the sum of the two numbers must be 6. So it must be one of the following:

$$(1, 5), (2, 4), (4, 2), (5, 1).$$

If the pair contains the same number, then the number must be a divisor of 6. So we need to check $(1, 1), (2, 2), (3, 3)$. $(1, 1)$ does not work because there are only 4 steps in total. Both $(2, 2)$ and $(3, 3)$ work. So in this case there are $4 + 2 = 6$ outcomes. The answer is $11 + 6 = 17$. And the probability that one can move 6 steps would be $\frac{17}{36}$, which is almost one half.

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Theorem

Suppose a finite set A *equals the union of* k *distinct mutually disjoint subsets* $A_1, A_2, \ldots, A_k$. *Then*

$$N(A) = N(A_1) + N(A_2) + \cdots + N(A_k).$$

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Suppose a finite set A equals the union of k distinct mutually disjoint subsets $A_1, A_2, \dots, A_k$. Then

$$N(A) = N(A_1) + N(A_2) + \cdots + N(A_k).$$

In other words, if an operation consists of one step which could be done in k cases, then the total number of ways is the sum of the number of ways in those k cases.

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By the addition rule, the total number of passwords equals the number of passwords of length 1*, plus the number of passwords of length* 2*, plus the number of passwords of length* 3*.*

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Solution

By the addition rule, the total number of passwords equals the number of passwords of length 1*, plus the number of passwords of length* 2*, plus the number of passwords of length* 3*. Note that by the multiplication rule, the number of passwords of length* k *is* 26^k*. So the answer is*

$$26 + 26^2 + 26^3 = 18,278.$$

The difference rule

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Remark

The difference rule provides an alternative way for counting: one can count the cardinality of a bigger set first, then subtract the number of excessive elements.

Exercise: marked elements in a sequence

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Five people are assigned to five seats in a row in a cinema. There is a couple among the five people. How many assignments make the couple to sit on nonadjacent seats?

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Solution

Each assignment is a permutation of the 5 *people, so total number of assignment is* $5! = 120$*. There are many ways to assign the couple to nonadjacent seats, so we can consider the opposite case - the couple are assigned to adjacent seats.*

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Five people are assigned to five seats in a row in a cinema. There is a couple among the five people. How many assignments make the couple to sit on nonadjacent seats?

Solution

Each assignment is a permutation of the 5 *people, so total number of assignment is* $5! = 120$. *There are many ways to assign the couple to nonadjacent seats, so we can consider the opposite case - the couple are assigned to adjacent seats. There are* 4 *pairs of adjacent seats* $(12, 23, 34, 45)$, *so we apply the addition rule. If the couple are assigned to any of these pairs of adjacent seats, there are still* $2! = 2$ *ways to assign the couple, and* $(5-2)! = 3! = 6$ *ways to assign other people. By the multiplication rule, there are* $2 \cdot 6 = 12$ *ways in each case. So there are* $12 \cdot 4$ *assignments such that the couple are assigned to adjacent seats. By the difference rule, the answer is* $120 - 48 = 72$.

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Recall the backgammon example. In the case direct 6, *it seems that we applied the addition rule as well: partition into first roll being* 6 *and second roll being* 6. *But the answer is not* $6 + 6 = 12$.

Dealing with repeated elements

Note that in order to apply the addition rule, the subsets must be mutually disjoint. While in practice this is not always true.

Remark

Recall the backgammon example. In the case direct 6, it seems that we applied the addition rule as well: partition into first roll being 6 and second roll being 6. But the answer is not $6+6=12$. The reason is that the two subsets are not disjoint, the pair could be $(6,6)$! While in this case it is quite simple, as this pair is counted twice, we can simply minus 1 and we are done. And we do have a formula for the general case.

The inclusion/exclusion rule for two and three sets

Theorem

If A, B, C are any finite sets, then

$$N(A \cup B) = N(A) + N(B) - N(A \cap B)$$

and

$$\begin{aligned} N(A \cup B \cup C) = {} & N(A) + N(B) + N(C) \\ & - N(A \cap B) - N(A \cap C) - N(B \cap C) \\ & + N(A \cap B \cap C). \end{aligned}$$

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Remark

There is a general formula for n sets, while we only use the above formulas in most cases.

Exercise: multiples

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How many integers from 1 *through* 100 *are multiples of* 3 *or multiples of* 5*? How many integers from* 1 *through* 100 *are neither multiples of* 3 *or multiples of* 5*?*

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Solution

Let m_k be the set of multiples of k from 1 through 100. We need to compute $N(m_3 \cup m_5)$. By the inclusion/exclusion rule, it equals $N(m_3) + N(m_5) - N(m_3 \cap m_5)$. Note that an integer is a multiple of both 3 and 5 if and only if it is a multiple 15. In addition, $N(m_k) = \lfloor \frac{100}{k} \rfloor$, so

$$N(m_3 \cup m_5) = \left\lfloor \frac{100}{3} \right\rfloor + \left\lfloor \frac{100}{5} \right\rfloor - \left\lfloor \frac{100}{15} \right\rfloor = 33 + 20 - 6 = 47.$$

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By the difference rule, the answer is $N((m_3 \cup m_5)^c) = 100 - N(m_3 \cup m_5) = 100 - 47 = 53.$

Exercise: European countries

Exercise

Currently, European Union (EU) consists of 28 *countries. The Schengen Area consists of* 26 *countries. Iceland, Liechtenstein, Norway and Switzerland are the only* 4 *countries that belong to the Schengen Area but are not members of the EU. How many countries belong to at least one of the EU and the Schengen Area?*

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Currently, European Union (EU) consists of 28 countries. The Schengen Area consists of 26 countries. Iceland, Liechtenstein, Norway and Switzerland are the only 4 countries that belong to the Schengen Area but are not members of the EU. How many countries belong to at least one of the EU and the Schengen Area?

Solution

We know that $N(EU) = 28$, $N(Schengen) = 26$ *and* $N(Schengen - EU) = 4$. *Since for any two sets* A, B, $A = (A \cap B) \cup (A - B)$. *Then* $N(Schengen \cap EU) = 26 - 4 = 22$. *And by the inclusion/exclusion rule,*

$$\begin{aligned} N(Schengen \cup EU) = {} & N(EU) + N(Schengen) \\ & - N(Schengen \cap EU) = 28 + 26 - 22 = 32. \end{aligned}$$

The principle

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A function from one finite set to a smaller finite set cannot be one-to-one: There must be a least two elements in the domain that have the same image in the co-domain.

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Theorem (The pigeonhole principle)

A function from one finite set to a smaller finite set cannot be one-to-one: There must be a least two elements in the domain that have the same image in the co-domain.
In other words, if n objects are put into m sets and $n > m$, then at least one set must contain two or more objects.

Exercise: birthday

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There are 33 *students in this class. Show that there must be two students whose birthdays are in the same month of the year.*

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Proof.

There are 12 months of a year and 33 students, since $33 > 12$, by the pigeonhole principle, there must be two students whose birthdays are in the same month of the year. □

Generalized Pigeonhole Principle

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If n objects are put into m sets, then at least one set must contain at least $\left\lceil \frac{n}{m} \right\rceil$ objects.

Theorem (Infinite version)

If infinitely many objects are put into finitely many sets, then at least one set must contain infinitely many objects.

Exercise: number of picks needed to ensure a result

Exercise

Suppose in an election there are 10 *candidates, and each voter votes for exactly one of them. What is the smallest number of voters such that no matter how they vote, there is always a candidate who gets at least* 5 *votes?*

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Solution

Suppose there are n *voters, by the generalized pigeonhole principle, there is always a candidate who gets at least* $\left\lceil \frac{n}{10} \right\rceil$ *votes. So if this number is* 5, n *is good. Note that*

$$\left\lceil \frac{40}{10} \right\rceil = 4, \left\lceil \frac{41}{10} \right\rceil = 5.$$

So 41 *voters is enough. While* 40 *voters may not work if each candidate gets exactly* 4 *votes. So the answer is* 41.

Exercise: large class size

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Solution

Suppose there are x discussion sections, by the generalized pigeonhole principle, there is a section with at least $\left\lceil \frac{300}{x} \right\rceil$ students. So $\frac{300}{x} \leq 28$, which means $\frac{300}{x} \leq 28$ and $x \geq \frac{300}{28} = 10 + \frac{20}{28}$. Since x must be an integer, the minimal value of x is 11.