Math 325K - Lecture 26
Section 9.5 & 9.6

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Outline

- r-combinations.
- Pascal's formula.
- The binomial theorem.

Unordered selections

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- *One can select* 3 *toppings from a list on a large pizza.*
- *The coach can select any* 5 *players from the team roster to start in a basketball game.*
- *In a sweepstake, UFCU selects* 10 *customers to receive the same prize.*

r-combinations

Definition

*Let n and r be nonnegative integers with $r \leq n$. An r-**combination** of a set of n elements is a subset of r of the n elements. As indicated in Section 5.1, the symbol*

$$\binom{n}{r},$$

which is read "n choose r", denotes the number of subsets of size r (r-combinations) that can be chosen from a set of n elements.

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Example

$$\binom{n}{1} = n, \binom{n}{n} = 1.$$

The value of $\binom{n}{r}$

Like the r-permutations, we would like to find a formula for the value of $\binom{n}{r}$. Recall that $P(n,r) = \frac{n!}{(n-r)!}$.

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What are the 2-permutations and the 2-combinations of $\{1,2,3,4\}$?

Solution

There are $4 \cdot 3 = 12$ 2-permutations:

$$12, 13, 14, 21, 23, 24, 31, 32, 34, 41, 42, 43.$$

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There are $4 \cdot 3 = 12$ 2-permutations:

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And there are 6 2-combinations, each appears twice above:

$$\{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}.$$

A direct formula of $\binom{n}{r}$

Theorem

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Proof.

There are $\frac{n!}{(n-r)!}$ r-permutations. For each r-combination, how many r-permutations does it correspond to?

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Proof.

There are $\frac{n!}{(n-r)!}$ r-permutations. For each r-combination, how many r-permutations does it correspond to? Since the r elements are fixed, they are just the r-permutations on r elements, which means the number is $r!$ So each r-combination appears in $r!$ r-permutations. And thus

$$\binom{n}{r} = \frac{n!}{(n-r)!}/r! = \frac{n!}{r!(n-r)!}.$$

□

Exercise: Powerball lottery

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One buys a Powerball lottery ticket as follows: select five distinct unordered numbers from 1 *to* 69 *for the white balls, then select one number from* 1 *to* 26 *for the red Powerball. There is only one winning number of* 5 *white balls and* 1 *red ball for the top prize. How many possible outcomes of the winning numbers?*

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Solution

To get a combination, there are two steps: choose five white balls and one red ball. So we apply the multiplication rule first. For the white balls, they are a 5 *selection among* 69 *numbers, so the number of ways is* $\binom{69}{5}$*;*

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One buys a Powerball lottery ticket as follows: select five distinct unordered numbers from 1 to 69 for the white balls, then select one number from 1 to 26 for the red Powerball. There is only one winning number of 5 white balls and 1 red ball for the top prize. How many possible outcomes of the winning numbers?

Solution

To get a combination, there are two steps: choose five white balls and one red ball. So we apply the multiplication rule first. For the white balls, they are a 5 selection among 69 numbers, so the number of ways is $\binom{69}{5}$; for the red ball, it is a single choice from the 26 numbers, so there are 26 choices. The answer is

$$\binom{69}{5} \cdot 26 = 292,201,338.$$

Properties of $\binom{n}{r}$

Recall we have mentioned the following property of $\binom{n}{r}$:

Proposition

For nonnegative integers $n \geq r$, we have

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Fix n. The maximum of $\binom{n}{r}$ is obtained at which integer r?

Proposition

For any positive integer n and integer $0 \leq r \leq n$, we have

$$\binom{n}{r} \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

Pascal's formula

A natural question about $\binom{n}{r}$ is: how to compute it? We know that factorials work. However, it is a very inefficient method. The reason is apparent: many factors will be canceled out in the end, so to compute the products in the factorials seems unnecessary.

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Theorem (Pascal's formula)

For nonnegative integers n, r with $n+1 \geq r$, we have

$$\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}.$$

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Theorem (Pascal's formula)

For nonnegative integers n, r with $n+1 \geq r$, we have

$$\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}.$$

Remark

This is a recursive relation because all $\binom{n}{\cdot}$ values give all $\binom{n+1}{\cdot}$.

The proof of Pascal's formula

Proof.

$$\binom{n}{r-1} + \binom{n}{r} = \frac{n!}{(r-1)!(n-r+1)!} + \frac{n!}{r!(n-r)!}$$
$$= \frac{n! \cdot r}{r!(n-r+1)!} + \frac{n! \cdot (n-r+1)}{r!(n-r+1)!}$$
$$= \frac{n! \cdot [r + (n-r+1)]}{r!(n-r+1)!} = \frac{n! \cdot (n+1)}{r!(n-r+1)!}$$
$$= \frac{(n+1)!}{r!(n-r+1)!} = \binom{n+1}{r}.$$

□

A proof by counting

There is an alternative proof by counting.

Proof.

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$\binom{n+1}{r}$ is the number of r-combination among integers 1 through $n+1$. Then there are two disjoint cases: 1 is in the combination or not. If the number 1 is in the combination, then the remaining $r-1$ selected numbers form an $r-1$-combination among integers 2 through $n+1$. The number of such combinations is $\binom{n}{r-1}$; if the number 1 is not in the combination, then all r selected numbers are among integers 2 through $n+1$. The number of such combinations is $\binom{n}{r}$. By the addition rule, the total number of such combinations is the sum of $\binom{n}{r-1}$ and $\binom{n}{r}$. □

Pascal's triangle

The **Pascal's triangle** is a triangular array of numbers such that the n-th row are just the numbers

$$\binom{n-1}{0}, \binom{n-1}{1}, \binom{n-1}{2}, \cdots, \binom{n-1}{n-1}.$$

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							1							
						1		1						
					1		2		1					
				1		3		3		1				
			1		4		6		4		1			
		1		5		10		10		5		1		
	1		6		15		20		15		6		1	
1		7		21		35		35		21		7		1

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			1		4		6		4		1			
		1		5		10		10		5		1		
	1		6		15		20		15		6		1	
1		7		21		35		35		21		7		1

Remark

French mathematician Pascal studied it in the 17th century, while scholar from India, Iran, China and Italy also studied it earlier.

Exercise: next row in the Pascal's triangle

Exercise

The 8-th row of the Pascal's triangle has numbers

$$1, 7, 21, 35, 35, 21, 7, 1.$$

Find the numbers in the 9-th row of the Pascal's triangle.

Exercise: next row in the Pascal's triangle

Exercise

The 8-th row of the Pascal's triangle has numbers

$$1, 7, 21, 35, 35, 21, 7, 1.$$

Find the numbers in the 9-th row of the Pascal's triangle.

Solution

The numbers are

$$1, 1+7, 7+21, 21+35, 35+35, \cdots$$

which are

$$1, 8, 28, 56, 70, 56, 28, 8, 1.$$

Motivation

Definition

In algebra, a sum of two terms, such as $a + b$, is called a **binomial**.

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Remark

Except for the single terms, binomials are the simplest expressions, so we want to study their arithmetic operations, especially their product.

Exercise: expand a power of binomial

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Expand the product $(a+b)^4$.

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Solution

$$\begin{aligned}(a+b)^4 &= \left[(a+b)^2\right]^2 \\ &= \left[a^2+2ab+b^2\right]^2 \\ &= \left(a^4+2a^3b+a^2b^2\right)+\left(2a^3b+4a^2b^2+2ab^3\right) \\ &+ \left(a^2b^2+2ab^3+b^4\right) \\ &= a^4+4a^3b+6a^2b^2+4ab^3+b^4.\end{aligned}$$

The binomial theorem

The coefficients in the expansion of $(a+b)^4$ are the numbers in Pascal's triangle. This is not a coincidence.

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For any real numbers a, b and nonnegative integer n,

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}.$$

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Theorem (The binomial theorem)

For any real numbers a, b and nonnegative integer n,

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}.$$

Proof.

Each term in the expansion is of the form $a^k b^{n-k}$. For each k, how many such terms are there? To get such a term we need exactly k copies of a and $n-k$ copies of b, which means we need to choose k parentheses for a among the n parentheses. So this number is $\binom{n}{k}$. □

Binomial coefficients

Remark

Because of the binomial theorem, the numbers $\binom{n}{r}$ are also called **binomial coefficients**. *In LaTeX, the symbol is typed by*

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\binom{n}{r}
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Exercise: applications of the binomial theorem

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Show that for every positive integer n,

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Proof.

Plug-in $a = b = 1$ in the binomial theorem, the left hand side becomes 2^n and the right hand side becomes $\sum_{k=0}^{n} \binom{n}{k}$. □