

Math 325K - Lecture 26

Section 9.5 & 9.6

Bo Lin

December 6th, 2018

Outline

- r -combinations.
- Pascal's formula.
- The binomial theorem.

Unordered selections

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Example

- *One can select 3 toppings from a list on a large pizza.*
- *The coach can select any 5 players from the team roster to start in a basketball game.*
- *In a sweepstake, UFCU selects 10 customers to receive the same prize.*

r -combinations

Definition

Let n and r be nonnegative integers with $r \leq n$. An **r -combination** of a set of n elements is a subset of r of the n elements. As indicated in Section 5.1, the symbol

$$\binom{n}{r},$$

which is read " n choose r ", denotes the number of subsets of size r (r -combinations) that can be chosen from a set of n elements.

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Example

$$\binom{n}{1} = n, \binom{n}{n} = 1.$$

The value of $\binom{n}{r}$

Like the r -permutations, we would like to find a formula for the value of $\binom{n}{r}$. Recall that $P(n, r) = \frac{n!}{(n-r)!}$.

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What are the 2-permutations and the 2-combinations of $\{1, 2, 3, 4\}$?

Solution

There are $4 \cdot 3 = 12$ 2-permutations:

12, 13, 14, 21, 23, 24, 31, 32, 34, 41, 42, 43.

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There are $4 \cdot 3 = 12$ 2-permutations:

12, 13, 14, 21, 23, 24, 31, 32, 34, 41, 42, 43.

And there are 6 2-combinations, each appears twice above:

$\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}$.

A direct formula of $\binom{n}{r}$

Theorem

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Proof.

There are $\frac{n!}{(n-r)!}$ r -permutations. For each r -combination, how many r -permutations does it correspond to?

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Proof.

There are $\frac{n!}{(n-r)!}$ r -permutations. For each r -combination, how many r -permutations does it correspond to? Since the r elements are fixed, they are just the r -permutations on r elements, which means the number is $r!$ So each r -combination appears in $r!$ r -permutations. And thus

$$\binom{n}{r} = \frac{n!}{(n-r)!} / r! = \frac{n!}{r!(n-r)!}.$$



Exercise: Powerball lottery

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One buys a Powerball lottery ticket as follows: select five distinct unordered numbers from 1 to 69 for the white balls, then select one number from 1 to 26 for the red Powerball. There is only one winning number of 5 white balls and 1 red ball for the top prize. How many possible outcomes of the winning numbers?

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Solution

To get a combination, there are two steps: choose five white balls and one red ball. So we apply the multiplication rule first. For the white balls, they are a 5 selection among 69 numbers, so the number of ways is $\binom{69}{5}$;

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Solution

To get a combination, there are two steps: choose five white balls and one red ball. So we apply the multiplication rule first. For the white balls, they are a 5 selection among 69 numbers, so the number of ways is $\binom{69}{5}$; for the red ball, it is a single choice from the 26 numbers, so there are 26 choices. The answer is

$$\binom{69}{5} \cdot 26 = 292,201,338.$$

Properties of $\binom{n}{r}$

Recall we have mentioned the following property of $\binom{n}{r}$:

Proposition

For nonnegative integers $n \geq r$, we have

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Fix n . The maximum of $\binom{n}{r}$ is obtained at which integer r ?

Proposition

For any positive integer n and integer $0 \leq r \leq n$, we have

$$\binom{n}{r} \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

Pascal's formula

A natural question about $\binom{n}{r}$ is: how to compute it? We know that factorials work. However, it is a very inefficient method. The reason is apparent: many factors will be canceled out in the end, so to compute the products in the factorials seems unnecessary.

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Theorem (Pascal's formula)

For nonnegative integers n, r with $n + 1 \geq r$, we have

$$\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}.$$

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Remark

This is a recursive relation because all $\binom{n}{\cdot}$ values give all $\binom{n+1}{\cdot}$.

The proof of Pascal's formula

Proof.

$$\begin{aligned}\binom{n}{r-1} + \binom{n}{r} &= \frac{n!}{(r-1)!(n-r+1)!} + \frac{n!}{r!(n-r)!} \\ &= \frac{n! \cdot r}{r!(n-r+1)!} + \frac{n! \cdot (n-r+1)}{r!(n-r+1)!} \\ &= \frac{n! \cdot [r + (n-r+1)]}{r!(n-r+1)!} = \frac{n! \cdot (n+1)}{r!(n-r+1)!} \\ &= \frac{(n+1)!}{r!(n-r+1)!} = \binom{n+1}{r}.\end{aligned}$$



A proof by counting

There is an alternative proof by counting.

Proof.

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Pascal's triangle

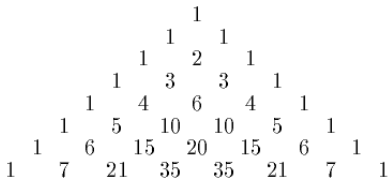
The **Pascal's triangle** is a triangular array of numbers such that the n -th row are just the numbers

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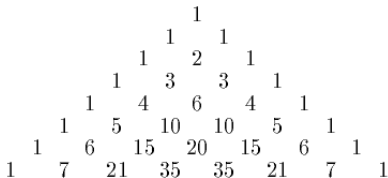
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Remark

French mathematician Pascal studied it in the 17th century, while scholar from India, Iran, China and Italy also studied it earlier.

Exercise: next row in the Pascal's triangle

Exercise

The 8-th row of the Pascal's triangle has numbers

$$1, 7, 21, 35, 35, 21, 7, 1.$$

Find the numbers in the 9-th row of the Pascal's triangle.

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$$1, 7, 21, 35, 35, 21, 7, 1.$$

Find the numbers in the 9-th row of the Pascal's triangle.

Solution

The numbers are

$$1, 1 + 7, 7 + 21, 21 + 35, 35 + 35, \dots$$

which are

$$1, 8, 28, 56, 70, 56, 28, 8, 1.$$

Motivation

Definition

*In algebra, a sum of two terms, such as $a + b$, is called a **binomial**.*

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In algebra, a sum of two terms, such as $a + b$, is called a **binomial**.

Remark

Except for the single terms, binomials are the simplest expressions, so we want to study their arithmetic operations, especially their product.

Exercise: expand a power of binomial

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Expand the product $(a + b)^4$.

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Solution

$$\begin{aligned}(a + b)^4 &= [(a + b)^2]^2 \\ &= [a^2 + 2ab + b^2]^2 \\ &= (a^4 + 2a^3b + a^2b^2) + (2a^3b + 4a^2b^2 + 2ab^3) \\ &\quad + (a^2b^2 + 2ab^3 + b^4) \\ &= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4.\end{aligned}$$

The binomial theorem

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$$(a + b)^n = \sum_{k=0}^n \binom{n}{k} a^k b^{n-k}.$$

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Theorem (The binomial theorem)

For any real numbers a, b and nonnegative integer n ,

$$(a + b)^n = \sum_{k=0}^n \binom{n}{k} a^k b^{n-k}.$$

Proof.

Each term in the expansion is of the form $a^k b^{n-k}$. For each k , how many such terms are there? To get such a term we need exactly k copies of a and $n - k$ copies of b , which means we need to choose k parentheses for a among the n parentheses. So this number is $\binom{n}{k}$. □

Binomial coefficients

Remark

*Because of the binomial theorem, the numbers $\binom{n}{r}$ are also called **binomial coefficients**. In \LaTeX , the symbol is typed by*

`\binom{n}{r}`

Exercise: applications of the binomial theorem

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Show that for every positive integer n ,

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Proof.

Plug-in $a = b = 1$ in the binomial theorem, the left hand side becomes 2^n and the right hand side becomes $\sum_{k=0}^n \binom{n}{k}$. □