Math 325K - Lecture 3
Section 2.2 Conditional statements

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September 6th, 2018

Outline

- The conditional operator $\rightarrow$.
- The related forms of $\rightarrow$.
- The biconditional operator $\leftrightarrow$.

The pattern of 'if ... then ...'

When one makes a logical reasoning, it is very common to reason from a premise to a conclusion. In other words, if is of the pattern

$$\text{if } p, \text{ then } q.$$

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if p, then q.

Since this pattern is so common and useful, we define a logical operator for it.

The operator $\to$

Definition

Given statements p and q, the **conditional** *of q by p is the statement 'if p then q', denoted by $p \to q$. It is false if and only if p is true and q is false. We call p the* **hypothesis** *and call q the* **conclusion**.

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Remark

The truth table of the statement form $p \to q$ is the following:

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Caution: when the hypothesis p is false

By definition, note that when the hypothesis p is false, no matter whether q is true or false, the statement $p \to q$ is always true.

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By definition, note that when the hypothesis p is false, no matter whether q is true or false, the statement $p \rightarrow q$ is always true.

Remark

As a consequence, even if q and p are completely irrelevant, $p \rightarrow q$ is still true when p is false. Roughly speaking, a false hypothesis could imply anything.

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By definition, note that when the hypothesis p is false, no matter whether q is true or false, the statement $p \rightarrow q$ is always true.

Remark

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Remark

The operator $\rightarrow$ is called 'implies' by many people. But there are opinions that 'implies' does not make sense because q and p may be irrelevant as explained above.

Examples of conditional statements

Example

Are the following conditional statements true or false?

(a) *if* 12 *is divisible by* 4*, then it is divisible by* 2*.*

(b) *if* $3 = 2$*, then* $3 = 5$*.*

(c) *if M325K has an assignment due next Tuesday, then it also has an assignment due next Thursday.*

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(c) *if M325K has an assignment due next Tuesday, then it also has an assignment due next Thursday.*

Solution

	Hypothesis	*Conclusion*	*Conditional statement*
(a)	T	T	T
(b)	F	F	T
(c)	T	F	F

Example: write $\to$ using $\sim, \wedge, \vee$

Proposition

The statement forms $p \to q$ and $\sim p \vee q$ are logically equivalent.

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Proposition

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Proof.

It suffices to write the truth tables of the latter.

p	q	$\sim p$	$\sim p \vee q$
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

$\square$

Example: dividing into cases

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$s = (p \to r) \wedge (q \to r)$.

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Proof.

p	q	r	$p \vee q$	$p \to r$	$q \to r$	$(p \vee q) \to r$	s
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F
T	F	T	T	T	T	T	T
T	F	F	T	F	T	F	F
F	T	T	T	T	T	T	T
F	T	F	T	T	F	F	F
F	F	T	F	T	T	T	T
F	F	F	F	T	T	T	T

The negation of $p \to q$

Proposition

The negation of 'if p then q' is logically equivalent to 'p and not q'.

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Proof.

$\sim(p \to q) \equiv \sim(\sim p \vee q)$. By De Morgan's Laws, it is logically equivalent to $(\sim(\sim p)) \wedge \sim q \equiv p \wedge \sim q$. $\square$

Example of negation of conditional statements

Example

Write the negation of 'If Jane lives in Athens, then she lives in Greece'.

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Solution

The answer is 'Jane lives in Athens and she does not live in Greece'.

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Solution

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Remark

This is possible, as there are cities named Athens in Georgia, Ohio and Wisconsin.

The contrapositive of $p \rightarrow q$

Definition

The **contrapositive** *of a conditional statement form* $p \rightarrow q$ *is the statement form* $\sim q \rightarrow \sim p$.

The contrapositive of $p \to q$

Definition

The **contrapositive** *of a conditional statement form $p \to q$ is the statement form $\sim q \to \sim p$.*

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$$p \to q \equiv \sim q \to \sim p.$$

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Proof.

For both statement forms, they are false if and only if p is true and q is false. $\square$

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Proof.

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Remark

This result of logical equivalence is very fundamental, as it provides an alternative way to do reasoning.

Examples of contrapositive

Example

Write the contrapositive of the following conditional statements.

(a) *If the escalators of the RLM building are operating, then the elevators of this building do not stop by the* 5*th floor.*

(b) *All prime numbers are not even.*

(c) *If* R *is a relation from set* A *to set* B*, then* R *is a subset of* $A \times B$*.*

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(c) *If* R *is a relation from set* A *to set* B*, then* R *is a subset of* $A \times B$*.*

Solution

(a) if the elevators of the RLM building stop by the 5*th floor, then the escalators of the RLM building are not operating.*

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Solution

(a) if the elevators of the RLM building stop by the 5th floor, then the escalators of the RLM building are not operating.
(b) if a number is even, then it is not a prime number.

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(b) *All prime numbers are not even.*

(c) *If R is a relation from set A to set B, then R is a subset of $A \times B$.*

Solution

(a) if the elevators of the RLM building stop by the 5th floor, then the escalators of the RLM building are not operating.
(b) if a number is even, then it is not a prime number.
(c) if R is not a subset of the Cartesian product $A \times B$, then R is not a relation from A to B.

The converse and inverse of $p \to q$

Definition

The **converse** *of $p \to q$ is $q \to p$. And the* **inverse** *of $p \to q$ is $\sim p \to \sim q$.*

The converse and inverse of $p \rightarrow q$

Definition

The **converse** *of $p \rightarrow q$ is $q \rightarrow p$. And the* **inverse** *of $p \rightarrow q$ is $\sim p \rightarrow \sim q$.*

Corollary

The converse and the inverse of $p \rightarrow q$ are contrapositive to each other and thus they are logically equivalent.

Examples of the converse and the inverse

Example

Write the converse and the inverse of 'if a number is even, then it is not a prime number'.

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Converse: if a number is not a prime number, then it is even.

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Solution

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Are they true or false?

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Write the converse and the inverse of 'if a number is even, then it is not a prime number'.

Solution

Converse: if a number is not a prime number, then it is even.
Inverse: if a number is not even, then it is a prime number.

Are they true or false?

Remark

They are both false as 9 *is neither even nor a prime number.*

The statement form $p \leftrightarrow q$

So far we have seen a lot of logically equivalent statement forms. How to characterize the relationship between them? We introduce another binary logical operator.

Definition

Given statements p and q, the **biconditional** *of p and q is the statement 'p if and only if q', denoted by $p \leftrightarrow q$. It is true if and only if p and q have the same truth value.*

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Remark

We also say iff for 'if and only if'.

The truth table of biconditional

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Remark

The priority of the five operators is

$$\sim \textit{ higher than } \wedge, \vee \textit{ higher than } \to, \leftrightarrow .$$

When operators of the same priority appear together, we need to add parentheses.

Example: write $\leftrightarrow$ using $\rightarrow$

Proposition

$$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p).$$

Example: write $\leftrightarrow$ using $\to$

Proposition

$$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p).$$

Proof.

p	q	$p \to q$	$q \to p$	$p \leftrightarrow q$	$(p \to q) \wedge (q \to p)$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	T	F	F	F
F	F	T	T	T	T

$\square$

Example: write $\leftrightarrow$ using $\to$

Proposition

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Proof.

p	q	$p \to q$	$q \to p$	$p \leftrightarrow q$	$(p \to q) \wedge (q \to p)$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	T	F	F	F
F	F	T	T	T	T

$\square$

Remark

As a consequence, we can also write $\leftrightarrow$ *using* $\sim, \wedge, \vee$ *only.*

A property of biconditional

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For two statement forms p and q, they are logically equivalent iff $p \leftrightarrow q$ is a tautology.

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Proof.

p and q are logically equivalent iff they have the same truth value for every combination of truth values of their common statement variables iff $p \leftrightarrow q$ is always true iff $p \leftrightarrow q$ is a tautology. □

Necessary and sufficient conditions

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Definition

Let r and s be statements.

- *'r is a necessary condition for s' means that 'if s then r', or $s \to r$ is true;*
- *'r is a sufficient condition for s' means that 'if r then s', or $r \to s$ is true;*
- *'r is a necessary and sufficient condition for s' means that 'r if and only if s', or $r \leftrightarrow s$ is true.*

Example: parse the statement for necessary and sufficient conditions

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Consider the statement 'if John is eligible to drive a vehicle, then he is at least 16 *years old'. Find necessary and sufficient conditions of the components in this statement.*

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Solution

Let d *be the statement 'John is eligible to drive a vehicle' and* s *be the statement 'John is at least sixteen years old'. Apparently,* s *is a necessary condition of* d*, while conversely,* d *is a sufficient condition of* s*.*

Example: converting necessary and sufficient conditions into 'if ... then ...' form

Example

Rewrite the following statements in the form 'if ... then ...':

(a) *Having a valid UT ID card is a sufficient condition that one can board a CapMetro bus without charge.*

(b) *Being a U.S. Citizen is a necessary condition that one can vote in a political election in the U.S.*

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Solution

(a) if one has a valid UT ID card, then one can board a CapMetro bus without charge.

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- (a) *Having a valid UT ID card is a sufficient condition that one can board a CapMetro bus without charge.*
- (b) *Being a U.S. Citizen is a necessary condition that one can vote in a political election in the U.S.*

Solution

(a) if one has a valid UT ID card, then one can board a CapMetro bus without charge.
(b) if one can vote in a political election in the U.S., then he/she is a U.S. Citizen.