Math 325K - Lecture 4
Section 2.3 Valid and invalid arguments

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Outline

- Arguments and argument forms.
- Examples of valid argument forms.
- Fallacies and contradiction rule.

Motivation

When we do reasoning, it may take multiple steps to reach a conclusion. It is very important to make sure that each step is correct. So we study the patterns of reasoning with multiple steps (statements).

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When we do reasoning, it may take multiple steps to reach a conclusion. It is very important to make sure that each step is correct. So we study the patterns of reasoning with multiple steps (statements).

Definition

An **argument** *is a sequence of statements, and an* **argument form** *is a sequence of statement forms. All statements in an argument and all statement forms in an argument form, except for the final one, are called* **premises**. *The final statement or statement form is called the* **conclusion**. *The symbol* $\therefore$, *which is read 'therefore', is normally placed just before the conclusion.*

Valid and invalid arguments

Definition

An argument form is called **valid** *if no matter what particular statements are substituted for the statement variables in its premises, if the resulting premises are all true, then the conclusion is also true. An argument is called* **valid** *if its form is valid.*

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Remark

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Remark

The definition of valid is consistent with our common sense about correct reasoning: if premises are true, then the conclusion must be true.

Remark

Equivalently, an argument form is valid if and only if 'the conjunction of its premises implies its conclusion' is a tautology.

Testing validity of argument forms

In the definition of validity of an argument form, we need to consider all possible combinations of the truth values of the statement variables in its premises, so a straight forward way to test validity is to write a truth table.

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Remark

The procedures of testing validity of an argument form:

1. *Identify the premises and conclusion of the argument form.*
2. *Construct a truth table showing the truth values of all the premises and the conclusion.*
3. *A row of the truth table in which all the premises are true is called a* **critical row**. *By definition, the argument form is valid if and only if the conclusion in every critical row is true.*

Example of an argument

Example

Consider the following argument form:

$$p \vee q$$
$$\sim q$$
$$\therefore p.$$

Is it valid?

Solution of example

Solution

We write the truth table:

		premise	premise	conclusion
p	q	$\sim q$	$p \vee q$	p
T	T	F	T	T
T	F	T	T	T
F	T	F	T	F
F	F	T	F	F

Solution of example

Solution

We write the truth table:

		premise	premise	conclusion
p	q	$\sim q$	$p \vee q$	p
T	T	F	T	T
T	F	T	T	T
F	T	F	T	F
F	F	T	F	F

The only critical row is the second row, and the conclusion there is true, so the argument form is valid.

Modus ponens

Now we introduce some basic valid argument forms that we can use in reasoning. The very first one and the most fundamental one is modus ponens.

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Definition

Modus ponens *is the following valid argument form:*

$$
\begin{aligned}
& p \\
& p \to q \\
\therefore\, & q.
\end{aligned}
$$

Remark

In Latin, 'modus ponens' means 'method of affirming'. And it is indeed an important way to affirm a statement.

Validity of modus ponens

Even though it is so straight forward, we still need to verify the validity of modus ponens. We can write its truth table

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Proof.

	premise	premise	conclusion
q	p	$p \to q$	q
T	T	T	T
T	F	T	T
F	T	F	F
F	F	T	F

The only critical row is the first row, and modus ponens is valid. $\square$

Modus tollens

Definition

Modus tollens *is the following valid argument form:*

$$p \to q$$
$$\sim q$$
$$\therefore \sim p.$$

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$$\therefore \sim p.$$

Remark

Since $p \to q$ *is logically equivalent to its contrapositive* $\sim q \to \sim p$*, its validity follows from modus ponens.*

Example of modus tollens

Example

What conclusion can you draw from the premises 'If a day is a weekday, then the canteen J2 is open for dinner on that day' and 'J2 is not open for dinner on the day with a football game'?

Example of modus tollens

Example

What conclusion can you draw from the premises 'If a day is a weekday, then the canteen J2 is open for dinner on that day' and 'J2 is not open for dinner on the day with a football game'?

Solution

Let p be 'the day with a football game is a weekday' and q be 'J2 is open for dinner on the day with a football game'. We have premises $p \to q$ and ~q, so the answer is ~p, which is 'the day with a football game is not a weekday'.

Rule of inference - generalization

Definition

The following valid argument forms are called **generalization***:*

$$\begin{array}{c} p \\ \therefore p \vee q. \end{array} \qquad\qquad \begin{array}{c} q \\ \therefore p \vee q. \end{array}$$

Rule of inference - generalization

Definition

The following valid argument forms are called **generalization***:*

$$\begin{array}{ll} p & q \\ \therefore p \vee q. & \therefore p \vee q. \end{array}$$

Remark

Its validity follows from the property of disjunction.

Rule of inference - specialization

Definition

The following valid argument forms are called **specialization***:*

$$\begin{array}{ll} p \wedge q \\ \therefore p. \end{array} \qquad \qquad \begin{array}{ll} p \wedge q \\ \therefore q. \end{array}$$

Rule of inference - specialization

Definition

The following valid argument forms are called **specialization***:*

$$\begin{array}{l} p \wedge q \\ \therefore p. \end{array} \qquad\qquad \begin{array}{l} p \wedge q \\ \therefore q. \end{array}$$

Remark

Its validity follows from the property of conjunction.

Example of reasoning

Example

What is the form of the following valid argument?

Jane is a math major

$\therefore$ *Jane is a math major or Jane is a computer science major*

Example of reasoning

Example

What is the form of the following valid argument?

$$\begin{aligned} &\text{Jane is a math major} \\ \therefore\ &\text{Jane is a math major or Jane is a computer science major} \end{aligned}$$

Solution

It is the form of generalization.

Rule of inference - elimination

Definition

The following valid argument forms are called **elimination***:*

$$\begin{array}{l} p \vee q \\ \sim q \\ \therefore p. \end{array} \qquad\qquad \begin{array}{l} p \vee q \\ \sim p \\ \therefore q. \end{array}$$

Rule of inference - elimination

Definition

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Remark

Its validity follows from the property of disjunction.

Rule of inference - transitivity

Definition

The following valid argument form is called **transitivity***:*

$$p \to q$$
$$q \to r$$
$$\therefore p \to r.$$

Rule of inference - transitivity

Definition

The following valid argument form is called **transitivity**:

$$p \to q$$
$$q \to r$$
$$\therefore p \to r.$$

Remark

To justify its validity, simply consider the case when both premises are true while the conclusion is false. Then p must be true and r must be false. By modus ponens of p and q, q must be true; By modus tollens of q and r, q must be false. So this case cannot happen and hence transitivity is valid.

Rule of inference - division into cases

Definition

The following valid argument form is called **division into cases***:*

$$\begin{aligned} & p \vee q \\ & p \to r \\ & q \to r \\ \therefore\; & r. \end{aligned}$$

Rule of inference - division into cases

Definition

The following valid argument form is called **division into cases***:*

$$p \vee q$$
$$p \to r$$
$$q \to r$$
$$\therefore r.$$

Remark

Suppose the premises are true and the conclusion r is false. By modus tollens of p and r, p must be false; and by modus tollens of q and r, q must be false; then $p \vee q$ cannot be true, so this case cannot happen and division into cases is valid.

Example: a more complex deduction

Example

In a morning, you realize that you don't have your glasses. You know the following statements are true:

a. *If I was reading the newspaper in the kitchen, then my glasses are on the kitchen table.*
b. *If my glasses are on the kitchen table, then I saw them at breakfast.*
c. *I did not see my glasses at breakfast.*
d. *I was reading the newspaper in the living room or I was reading the newspaper in the kitchen.*
e. *If I was reading the newspaper in the living room then my glasses are on the coffee table.*

Then where are your glasses?

Example: a more complex deduction

Remark

For convenience, let's denote the statements involved as follows:

- RK = *I was reading the newspaper in the kitchen.*
- GK = *My glasses are on the kitchen table.*
- SB = *I saw my glasses at breakfast.*
- RL = *I was reading the newspaper in the living room.*
- GC = *My glasses are on the coffee table.*

Example: a more complex deduction

Remark

Now the statements become:

(a) $RK \to GK$.

(b) $GK \to SB$.

(c) $\sim SB$.

(d) $RL \vee RK$.

(e) $RL \to GC$.

Example: a more complex deduction

Solution

Apply modus tollens to (b) and (c), we have

$$\sim GK. \tag{f}$$

Apply modus tollens to (a) and (f), we have

$$\sim RK. \tag{g}$$

Apply elimination to (d) and (g), we have

$$RL. \tag{h}$$

Finally apply modus ponens to (h) and (e), we have GC. So the conclusion is that your glasses are on the coffee table.

Fallacies

Definition

A **fallacy** *is an error in reasoning that results in an invalid argument.*

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Three common types of fallacies:

- using ambiguous premises;
- assuming what is to be proved;
- jumping to a conclusion without adequate grounds.

Converse error and inverse error

Definition

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$$p \to q$$
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Converse error and inverse error

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$$\therefore p.$$

Definition

The **inverse error** *is the following invalid argument form:*

$$p \to q$$
$$\sim p$$
$$\therefore \sim q.$$

Warning: interpretation of invalid argument

Question

If an argument is invalid, what can we say about it?

Warning: interpretation of invalid argument

Question

If an argument is invalid, what can we say about it?

Remark

An argument being valid means that there is some flaw in the process of reasoning. It does not imply anything about the truth values of any premise and the conclusion. An invalid argument could have true premises and true conclusions, and a valid argument could also have false premises and false conclusions.

Example: some arguments

Example

Are the following arguments valid or invalid?

a. *if John is carrying a gun, then there is a hard item in his luggage. The x-ray scanner finds a hard item in John's luggage. In conclusion, John is carrying a gun.*
b. *Dallas is the capital city of Texas and New York City is the capital city of the state of New York. In conclusion, Dallas is the capital city of Texas.*
c. 2 *is an even number or* 2 *is a prime number. And* 2 *is indeed an even number. In conclusion,* 2 *is a prime number.*

Example: some arguments

Solution

(a) has premises $p \to q$ and q, and conclusion p. It is the form of converse error, so (a) is invalid.

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Solution

(a) has premises $p \to q$ and q, and conclusion p. It is the form of converse error, so (a) is invalid.
(b) has premise $p \wedge q$ and conclusion p, which is of the form specialization. So (b) must be valid, even though both its premise and conclusion are false. (Actually, Austin and Albany are the capital cities of TX and NY).

Example: some arguments

Solution

(a) has premises $p \to q$ and q, and conclusion p. It is the form of converse error, so (a) is invalid.
(b) has premise $p \wedge q$ and conclusion p, which is of the form specialization. So (b) must be valid, even though both its premise and conclusion are false. (Actually, Austin and Albany are the capital cities of TX and NY).
(c) has premises $p \vee q$ and p. and conclusion q. When p is true and q is false, both premises are true but the conclusion is false. So (c) is invalid.

Example: some arguments

Solution

(a) has premises $p \to q$ and q, and conclusion p. It is the form of converse error, so (a) is invalid.
(b) has premise $p \wedge q$ and conclusion p, which is of the form specialization. So (b) must be valid, even though both its premise and conclusion are false. (Actually, Austin and Albany are the capital cities of TX and NY).
(c) has premises $p \vee q$ and p. and conclusion q. When p is true and q is false, both premises are true but the conclusion is false. So (c) is invalid.

Remark

When we test the validity of an argument, we only consider its form.

Sound arguments

For argument like (b) in the previous example, it may not be very useful. So we have the following definition to exclude them.

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Definition

An argument is called **sound** *if and only if it is valid and all its premises are true. An argument that is not sound is called* **unsound***.*

Sound arguments

For argument like (b) in the previous example, it may not be very useful. So we have the following definition to exclude them.

Definition

An argument is called **sound** *if and only if it is valid and all its premises are true. An argument that is not sound is called* **unsound**.

Remark

Validity is a property of argument forms and soundness is a property of arguments.

Rules of inference - contradiction rule

Proposition (contradiction rule)

If one can show that the assumption that statement p is false leads logically to a contradiction, then one can conclude that p is true.

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Remark

In symbols, we can write the following valid argument form:

$$\begin{array}{l} {\sim}p \to c, \text{ where } c \text{ is a contradiction} \\ \therefore p. \end{array}$$

Validity of the contradiction rule

Proof.

we write the truth table for the contradiction rule:

	contradiction		premise	conclusion
p	c	$\sim p$	$\sim p \to c$	p
T	F	F	T	T
F	F	T	F	F

The only critical row is the first row and the argument form is indeed valid. □

Summary of valid argument forms

We introduced the following valid argument forms that you can freely use them in homework and exams:

- modus ponens;
- modus tollens;
- generalization;
- specialization;
- elimination;
- transitivity;
- division into cases;
- contradiction rule.

HW in this section

Exercise 4, 11, 19, 28, 32, 42.