Math 325K - Lecture 5
Section 3.1 & 3.2
Predicates and quantified statements

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Outline

- Predicates and quantifiers.
- Universal and existential quantified statements.
- Other forms of quantified statements.

Motivation

Now we can do reasoning by valid argument forms. But there are a lot more complicated arguments. For example: all humans are mortal; Socrates is a human; $\therefore$ Socrates is mortal.

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Now we can do reasoning by valid argument forms. But there are a lot more complicated arguments. For example: all humans are mortal; Socrates is a human; $\therefore$ Socrates is mortal.
This should be a valid argument form, but it does not fit any patterns we introduced last time. In fact, our tools to deal with statements are not enough. And as a result, we need to introduce some new concepts and tools to study these arguments, which is called predicate logic.

Predicates

In grammar, the word predicate refers to the part of a sentence that gives information about the subject.

Definition

A **predicate** *is a sentence that contains a finite number of variables and becomes a statement when specific values are substituted for the variables. The domain of a predicate variable is the set of all values that may be substituted in place of the variable.*

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Remark

Usually we can interpret a predicate as a function whose co-domain consists of statements.

Examples of predicates as functions

Example

Let $S(x)$ be the predicate $x^2 > x$ with domain $\mathbb{R}$. Rewrite the following statements in sentences:

(a) $S(2)$;

(b) $\sim S(1) \wedge S(0)$.

Are they true or false?

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Solution

(a) is simply $2^2 > 2$ *and it is true. (b) is '(not* $1^2 > 1$*) and* $0^2 > 0$*', which is equivalently '*$1^2 \leq 1$ *and* $0^2 > 0$*', so it is false.*

Truth set

Suppose we have a predicate $P(x)$ with domain D, then for each $y \in D$, $P(y)$ is a specific statement and it is either true or false. Then we would like to know that for what elements $y \in D$, $P(y)$ is true?

Truth set

Suppose we have a predicate $P(x)$ with domain D, then for each $y \in D$, $P(y)$ is a specific statement and it is either true or false. Then we would like to know that for what elements $y \in D$, $P(y)$ is true?

Definition

If $P(x)$ is a predicate with domain D, the **truth set** *of $P(x)$ is the set of all elements of D that make $P(x)$ true when they are substituted for x. The truth set of $P(x)$ is denoted*

$$\{x \in D \mid P(x) \text{ is true } \}.$$

Quantifiers

In a sentence, even if we fix the subject and the predicate, there is still a twist: the number of subjects referred to? For example, the following sentences have very different meanings. As a result, we must consider the quantifiers too.

- all humans are mortal;
- some humans are mortal;
- one human is mortal;
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Definition

Quantifiers *are words that refer to quantities such as 'some' or 'all' and tell for how many elements a given predicate is true.*

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The **universal quantifier***, written as $\forall$ and read 'for all', refers to all elements in the domain.*

Definition

The **existential quantifier***, written as $\exists$ and read 'there exists/there exist', refers to at least one element in the domain.*

Universal statements

Definition

Let $Q(x)$ be a predicate and D the domain of x. A **universal statement** *is a statement of the form '$\forall x \in D, Q(x)$'. It is defined to be true if and only if $Q(x)$ is true for every x in D. It is defined to be false if and only if $Q(x)$ is false for at least one x in D. A value for x for which $Q(x)$ is false is called a* **counterexample** *to the universal statement.*

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Remark

One needs to specify the domain of universal statements. Sometimes this set may be implicitly given.

Examples of universal statements

Example

Let $R(x)$ be the predicate '$\sqrt{x}$ is a rational number' with domain $\mathbb{N}$ (why not $\mathbb{Z}$?), and $L(x)$ be the predicate '$x < 2x$' with domain $\mathbb{N}$. Rewrite the following universal statements in sentences and figure out whether they are true or false.

(a) $\forall x \in \mathbb{N}, R(x)$;

(b) $\forall x \in \mathbb{N}, L(x)$.

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(a) is 'the square root of all positive integers are rational numbers'. It is false as 2 is a counterexample.

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Let $R(x)$ *be the predicate '*$\sqrt{x}$ *is a rational number' with domain* $\mathbb{N}$ *(why not* $\mathbb{Z}$*?), and* $L(x)$ *be the predicate '*$x < 2x$*' with domain* $\mathbb{N}$*. Rewrite the following universal statements in sentences and figure out whether they are true or false.*

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Solution

(a) is 'the square root of all positive integers are rational numbers'. It is false as 2 *is a counterexample. (b) is 'for all positive integers* x*,* x *is less than* $2x$*', which is true.*

Existential statements

Definition

Let $Q(x)$ *be a predicate and* D *the domain of* x. *An* **existential statement** *is a statement of the form '*$\exists x \in D$ *such that* $Q(x)$*'. It is defined to be true if and only if* $Q(x)$ *is true for at least one* x *in* D. *It is defined to be false if and only if* $Q(x)$ *is false for all* x *in* D.

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Remark

Existential statements are somehow dual to universal statements, and we will introduce their connections in a while.

Examples of existential statements

Example

Rewrite the following existential statement in symbols and figure out whether it is true or false: there exists an integer that is prime and greater than 10.

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Solution

The answer is not unique:

- $\exists n \in \mathbb{Z}$ *such that* n *is prime and* $n > 10$;
- $\exists n \in \{x \in \mathbb{Z} \mid x > 10\}$ *such that* n *is prime;*
- $\exists n \in$ *the set of prime numbers such that* $n > 10$.

Since 11 *is prime and greater than* 10*, this existential statement is true.*

Universal conditional statements

Definition

A **universal conditional statement** *is of the form*

$$\forall x \in D, \text{ if } P(x) \text{ then } Q(x),$$

where $P(x), Q(x)$ *are predicates with domain* D.

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Remark

Let U *be the truth set of* $P(x)$*, then the above universal conditional statement is logically equivalent to*

$$\forall x \in U, Q(x).$$

Implicit quantification

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Write the following statement using symbols: if a person is a member of UT, then he/she has a UT EID. You may need to define predicates by yourselves.

Implicit quantification

As we have already seen, some statements are written in a way without quantifiers. However, we can equivalently rewrite them with $\forall$ or $\exists$.

Example

Write the following statement using symbols: if a person is a member of UT, then he/she has a UT EID. You may need to define predicates by yourselves.

Solution

Let H be the set of humans, $M(x)$ be the predicate 'x is a member of UT' with domain H and $E(x)$ be the predicate 'x has a UT EID' with domain H. The statement becomes

$$\forall x \in H, M(x) \to E(x).$$

Equivalent notations

Definition

Let $P(x)$ and $Q(x)$ be predicates with domain D. For convenience we write $P(x) \Rightarrow Q(x)$ for

$$\forall x \in D, P(x) \to Q(x).$$

And we write $P(x) \Leftrightarrow Q(x)$ for

$$\forall x \in D, P(x) \leftrightarrow Q(x).$$

Finding the truth value of quantified statements

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Finding the truth value of quantified statements

Given a universal or existential statement, how to find its truth value?
If the domain is finite, we can simply substitute the variable by each element in the domain. This approach is called **exhaustion** and it is guaranteed to work in this case.
But in mathematics, many sets are infinite, which means we need to consider more efficient way. There is one shortcut: one counterexample is enough to show a universal statement being false and one example is enough to show a existential statement being true.

Negation of universal statements

Proposition

The negation of a statement of the form

$$\forall x \in D, Q(x)$$

is logically equivalent to a statement of the form

$$\exists x \in D \text{ such that } {\sim}Q(x).$$

Symbolically, ${\sim}(\forall x \in D, Q(x)) \equiv (\exists x \in D, \text{ such that } {\sim}Q(x))$.

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Remark

As a corollary, universal statements and existential statements can be defined by each other plus negation.

Negation of existential statements

Similarly we have

Proposition

The negation of a statement of the form

$$\exists x \in D \text{ such that } Q(x)$$

is logically equivalent to a statement of the form

$$\forall x \in D, \sim Q(x).$$

Symbolically, $\sim(\exists x \in D \text{ such that } Q(x)) \equiv (\forall x \in D, \sim Q(x))$.

Negation of universal conditional statements

Proposition

The negation of a statement of the form

$$\forall x \in D, P(x) \to Q(x)$$

is logically equivalent to a statement of the form

$$\exists x \in D \text{ such that } {\sim}(P(x) \to Q(x)).$$

Symbolically,

$$\begin{aligned} &{\sim}(\forall x \in D, P(x) \to Q(x)) \\ \equiv &\exists x \in D \text{ such that } {\sim}(P(x) \to Q(x)). \end{aligned}$$

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Example: find the negations

Example

Write the formal (with symbols) negations of the following quantified statements:

(a) *All computer programs are finite.*

(b) *There is a computer program in the programming language $Lisp$.*

(c) *If a computer program has more than 100,000 lines, then it contains a bug.*

Example: find the negations

Solution

Let P be the set of all computer programs. $F(x)$ be the predicate 'x is finite'; $L(x)$ be the predicate 'x is in `Lisp`'; $T(x)$ be the predicate 'x has more than $100,000$ lines'; and $B(x)$ be be the predicate 'x contains a bug', all with domain P.

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Let P be the set of all computer programs. $F(x)$ be the predicate 'x is finite'; $L(x)$ be the predicate 'x is in `Lisp`*'; $T(x)$ be the predicate 'x has more than $100,000$ lines'; and $B(x)$ be be the predicate 'x contains a bug', all with domain P.*
(a) is $\forall x \in P, F(x)$. So the negation is $\exists x \in P$ such that $\sim F(x)$.

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(a) is $\forall x \in P, F(x)$. So the negation is $\exists x \in P$ such that $\sim F(x)$.

(b) is $\exists x \in P$ such that $L(x)$. So its negation is $\forall x \in P, \sim L(x)$.

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Solution

Let P *be the set of all computer programs.* $F(x)$ *be the predicate '*x *is finite';* $L(x)$ *be the predicate '*x *is in* `Lisp`*';* $T(x)$ *be the predicate '*x *has more than* $100,000$ *lines'; and* $B(x)$ *be be the predicate '*x *contains a bug', all with domain* P*.*

(a) is $\forall x \in P, F(x)$*. So the negation is* $\exists x \in P$ *such that* $\sim F(x)$*.*

(b) is $\exists x \in P$ *such that* $L(x)$*. So its negation is* $\forall x \in P, \sim L(x)$*.*

(c) is $\forall x \in P, T(x) \to B(x)$*. So its negation is* $\exists x \in P$ *such that* $\sim(T(x) \to B(x))$*, equivalently*

$$\exists x \in P \text{ such that } (T(x) \wedge \sim B(x)).$$

Connection with $\wedge$ and $\vee$

Let $P(x)$ be a predicate with a finite domain $D = \{x_1, x_2, \ldots, x_n\}$. Then we have

Proposition

The universal statement $\forall x \in D, P(x)$ is logically equivalent to

$$P(x_1) \wedge P(x_2) \wedge \cdots \wedge P(x_n).$$

And the existential statement $\exists x \in D$ such that $P(x)$ is logically equivalent to

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Remark

We don't have a similar result when D is infinite.

Vacuous truth of universal statements

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Solution

It is of the form $\forall x \in D, P(x) \to Q(x)$ *while for all* $x \in D$*,* $P(x)$ *is false. Then for each* $x \in D$*, the conditional statement* $P(x) \to Q(x)$ *is by default true, hence the universal conditional statement is also true. And this is called the vacuous truth of them.*

Other forms of universal conditional statements

Definition

Let $P(x), Q(x)$ *be predicates with domain* D *and* s *be the statement*

$$\forall x \in D, P(x) \to Q(x)$$

The contrapositive of s *is*

$$\forall x \in D, \sim Q(x) \to \sim P(x).$$

The converse of s *is*

$$\forall x \in D, Q(x) \to P(x).$$

And the inverse of s *is*

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Warning: $\forall$ and $\exists$ are not commutative

As we will discuss in detail next time, the quantifiers are not commutative in general.

Example

Let $G(x, y)$ be the binary predicate $x < y$ with domain $\mathbb{N} \times \mathbb{N}$. Are the following statements true or false?

(a) $\forall x \in \mathbb{N}, \exists y \in \mathbb{N}$ *such that* $G(x, y)$.

(b) $\exists y \in \mathbb{N}$ *such that* $\forall x \in \mathbb{N}, G(x, y)$.

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Solution

(a) is true, as given $x \in N$, we can always choose $y = x + 1$ to justify the existential statement.

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Solution

(a) is true, as given $x \in N$, we can always choose $y = x + 1$ to justify the existential statement.
(b) is false, as given $y \in N$, $x = y + 1$ is a counterexample to the universal statement, so for all y it is false. So is (b).

HW #2 - these sections

Section 3.1 Exercise 6, 17(b), 23(b), 28(a)(c).
Section 3.2 Exercise 2, 14, 23, 46.