Math 325K - Lecture 6
Section 3.3 & 3.4

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September 18th, 2018

Outline

- Multi-quantified statements.
- Validity of quantified arguments and argument forms.
- Rules of inference for quantified statements.

Interpretation of multi-quantified statements

Recall the last example in the previous lecture. Let $G(x, y)$ be the binary predicate $x < y$ with domain $\mathbb{N} \times \mathbb{N}$. We considered a statement $\forall x \in \mathbb{N}, \exists y \in \mathbb{N}$ such that $G(x, y)$. Here both $\forall$ and $\exists$ appear in the same statement.

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In fact, a quantified statement may contain any finite number of quantifiers. Multiple quantifiers would make the statement more complicated, while the way we deal with them is almost the same as before.

Remark

We parse the sentence with multiple quantifiers according to the order they appear.

The scope of quantifiers

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Here y is not free, because when it is introduced, we are already with the choice of x. In other words, only when x is chosen, is fixed, then we consider y. So y completely depends on x and it is **bound**.

Examples: translating to formal expressions

Example

Rewrite the following multi-quantified statements in symbols: let H be the set of humans and $T(x, y)$ is the predicate "x is taller than y".

(a) *There is a human who is taller than anyone else.*

(b) *Everyone is taller than someone else.*

(c) *For any two humans, one is taller than the other.*

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Solution

(a) $\exists x \in H$ *such that* $\forall y \in H, y \neq x \to T(x, y)$. *Alternatively,* $\exists x \in H$ *such that* $\forall y \in H - \{x\}, T(x, y)$.

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Solution

(a) $\exists x \in H$ *such that* $\forall y \in H, y \neq x \rightarrow T(x,y)$. *Alternatively,* $\exists x \in H$ *such that* $\forall y \in H - \{x\}, T(x,y)$.
(b) $\forall x \in H, \exists y \in H$ *such that* $y \neq x \wedge T(x,y)$.

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(b) $\forall x \in H, \exists y \in H$ *such that* $y \neq x \wedge T(x,y)$.
(c) $\forall x \in H, \forall y \in H - \{x\}, T(x,y) \vee T(y,x)$.

Negation of $\forall\exists$ statements

Consider a binary predicate $P(x, y)$ with domain $D \times E$.

Proposition

$$\begin{aligned} &\sim(\forall x \in D, \exists y \in E \textit{ such that } P(x, y)) \\ \equiv &\exists x \in D \textit{ such that } \forall y \in E, \sim P(x, y). \end{aligned}$$

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$$\equiv \exists x \in D \textit{ such that } \forall y \in E, \sim P(x, y).$$

Remark

There is a pattern about negations of quantified statements: negate all quantifiers and the actual statement in the end.

Negation of ∃∀ statements

Similarly we have

Proposition

$$\sim(\exists x \in D \textit{ such that } \forall y \in E, P(x,y))$$
$$\equiv \forall x \in D, \exists y \in E \textit{ such that } \sim P(x,y).$$

Remark

In the negation, y depends on x.

Example: negation of multi-quantified statements

Example

Write the negation of the following statement in sentence: for every positive integer x, there is a rational number whose cube is x. Is the negation true?

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Solution

The original statement is: $\forall x \in \mathbb{N}, \exists y \in \mathbb{Q}$ *such that* $y^3 = x$. *Its negation is:*

$$\exists x \in \mathbb{N} \text{ such that } \forall y \in \mathbb{Q}, y^3 \neq x.$$

The negation is true as $x = 2$ *is a counterexample.*

Definition of validity

Definition

To say that an argument form is **valid** *means the following: No matter what particular predicates are substituted for the predicate symbols in its premises, if the resulting premise statements are all true, then the conclusion is also true. An argument is called* **valid** *if and only if its form is valid.*

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Remark

Validity is still a property of arguments and argument forms, not of statements.

Using diagram

There is a difference in testing validity of quantified argument forms. Now we may have multiple predicates and they have different truth sets, then we can draw diagrams to illustrate these truth sets and find the result.

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Remark

This method is very useful for arguments forms involving conditinal statement forms.

Example: diagram

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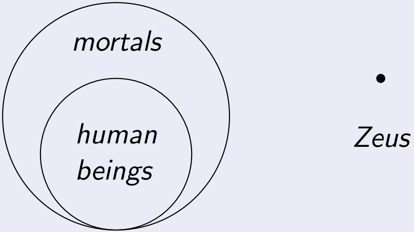
Now we use a diagram to show the validity of the Zeus argument.



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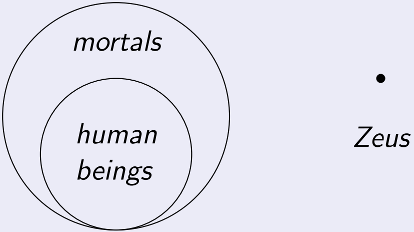


If the point representing Zeus is outside of the big circle, then it is also outside of the small circle, which means the argument is valid.

Exercise of diagram

Example

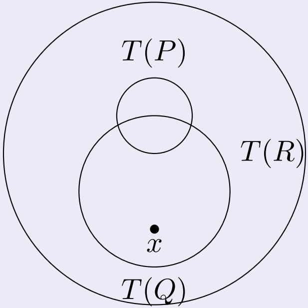
Use a diagram to test the validity of the following argument form:

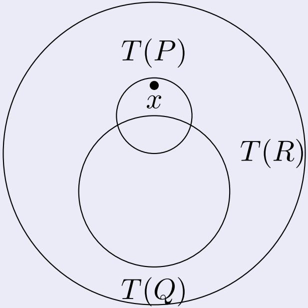
$$\forall x \in D, P(x) \to R(x)$$
$$\forall x \in D, Q(x) \to R(x)$$
$$\therefore \forall x \in D, P(x) \vee Q(x) \to R(x).$$

Exercise of diagram

Proof.

For convenience let $T(A)$ be the truth set of a predicate A.





□

Quantified form of converse and inverse errors

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The converse error is the following invalid argument form:

$$\forall x \in D, P(x) \to Q(x)$$
$$Q(a)(a \in D \text{ is a particular element})$$
$$\therefore P(a).$$

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Definition

The inverse error is the following invalid argument form:

$$\begin{aligned} &\forall x \in D, P(x) \to Q(x) \\ &{\sim} P(a)(a \in D \text{ is a particular element}) \\ \therefore &Q(a). \end{aligned}$$

Universal instantiation

Definition

The rule of **universal instantiation** *is: if some property is true of everything in a set, then it is true of any particular thing in the set. In symbols, it is the following valid argument form:*

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$$a \in D$$
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Remark

This rule is very useful, as it connects a quantified statement and a statement without quantifiers.

Example: Socrates

Example

The argument of Socrates "all humans are mortal; Socrates is a human; $\therefore$ Socrates is mortal" is valid and sound.

Universal modus ponens

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By universal instantiation we have $P(a) \to Q(a)$*, so its validity follows from modus ponens.*

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By universal instantiation we have $P(a) \to Q(a)$, so its validity follows from modus tollens.

Universal transitivity

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Universal transitivity *is the following valid argument form:*

$$\forall x \in D, P(x) \to Q(x)$$
$$\forall x \in D, Q(x) \to R(x)$$
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Remark

For each $x \in D$, $P(x) \to R(x)$ *is deduced from transitivity. As a result, this argument form is valid too.*

Examples: quantified arguments

Example

For each of the following quantified arguments, determine whether they are valid. If so, specify the type of valid argument form they belong to.

(a) *If an integer is even, then it equals twice some integer;* 100 *is an even integer;* $\therefore$ 100 *equals twice some integer* r.

(b) *All human beings are mortal; Zeus is not mortal;* $\therefore$ *Zeus is not a human being.*

(c) *If a NBA player committed* 6 *personal fouls in a game, he is not on the court anymore; LeBron James committed* 2 *personal fouls in a game;* $\therefore$ *LeBron James is on the court.*

(d) *Every integer* a *can also be written in the form* $\frac{a}{1}$*; for every integer* a*, the number* $\frac{a}{1}$ *is a rational number;* $\therefore$ *every integer is a rational number.*

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d. *Every integer* a *can also be written in the form* $\frac{a}{1}$*; for every integer* a*, the number* $\frac{a}{1}$ *is a rational number;* $\therefore$ *every integer is a rational number.*

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(a) This is an example of universal modus ponens and it is valid.

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(c) Let $F(x)$ be "x committed 6 fouls" and $C(x)$ be "x is on the court". The premises are $\forall x, F(x) \to C(x)$ and $\sim F(LBJ)$, the conclusion is $C(LBJ)$. It is invalid, as we only have $F(LBJ) \to C(LBJ)$, then it is an inverse error.

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HW #3 - these sections

Section 3.3 Exercise 9(b), 11(b)(e)(f), 17, 36, 41(c)(d).
Section 3.4 Exercise 4, 13, 15, 24, 34.