Math 2603 - Lecture 1
Chapter 0

Bo Lin

August 20th, 2019

Important things in syllabus

My information

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TA's Information

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Grade breakdown

- Homework 0%;
- Quizzes 30%, only best 10 grades count;
- Both midterm exams 20% each;
- Final exam 30%.

Classroom Policy

- Be respectful to other people;
- You can use your electronic devices for course-related purposes;
- Please mute your devices, as there's no audio or video contents in this course;
- Food and drink are allowed, but please keep noises to its minimum.

Course websites

Canvas: `https://gatech.instructure.com/courses/57052`
Piazza: `https://piazza.com/gatech/fall2019/math2603`
(Please signup to Piazza if you haven't)

Miscellaneous

- I have zero control of course registration, see syllabus for resources;
- Exams will be graded using Gradescope, please try not to use pencils in exams;
- Various resources for your academic and personal needs, see syllabus for details.

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A **statement** *or* **proposition** *is a sentence that is true or false but not both.*

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Remark

If one cannot tell whether the sentence is true or false, or its truthfulness depends on the value of some variable, then it is not a statement.

Examples of statements and non statements

Example

Are the following sentences statements?

(a) *The square of* 3 *is* 10.

(b) $a + b > 0$.

(c) *This Math 2603 class has* 2 *midterm exams and it has* 1 *final exam.*

(d) *The weather today is hot.*

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(c) *This Math 2603 class has* 2 *midterm exams and it has* 1 *final exam.*

(d) *The weather today is hot.*

Solution

(a), (c) are statements where (c) is true, (a) is false. (b) is not a statement, because its truthfulness depends on the values of variables a *and* b*. (d) is not a statement, as 'hot' is not defined and ambiguous.*

Compound statements

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There are several types of compound statements:

1. conjunction ("and");
2. disjunction ("or");
3. implication;
4. negation;
5. double implication ("equivalence").

Conjunction ("and")

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Given statements p and q, the **conjunction** *of p and q is the statement 'p and q', denoted by $p \wedge q$ (LaTeX symbol \wedge).*

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Confusion with English sentences

Remark

In English, some sentences with "or" are different. For example, "the terrorist is dead or alive". In this case, the two statements cannot both be true. In other words, it is of the structure "either...or...". This is the **exclusive or***. In our course, "or" always means* **inclusive or***, where at least one statement is true.*

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There are other words related to the relationships of statements.

Remark

The word 'but' serves as 'and' in general. For example, consider the compound statement 'the building is tall but it has no elevator'. If p denotes 'the building is tall' and q denotes 'the building has no elevator', the statement may be rephrased as $p \wedge q$.

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Remark

Consistent with our common sense, when p is true, $\neg p$ must be false; and when p is false, $\neg p$ must be true.

Implication

Many mathematical statements are of the form "p implies q", they are called implications.

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Remark

Maybe it is a little counterintuitive that $p \to q$ is false only when p is true and q is false!

Implication - what happens if the assumption is false

Let's analyze the following statement:
if $2 + 2 = 3$, then tomorrow is raining.
Is it true or false?

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Remark

Here the problem is that $2+2 \neq 3$. So the assumption does not hold. Then whether tomorrow is raining or not no longer matters. In either case, there is no violation or contradiction in this statement, so we don't have a justification to claim that it is false. And thus we define that it is true.

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Remark

In math and logic, fallacy implies everything.

Double implication

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Remark

$p \leftrightarrow q$ *is true when p and q are both true or both false.*

Examples of compound statements

Example

Let h be 'John is healthy', w be 'John is wealthy', and s be 'John is smart'. Figure out the type of the following compound statements and rewrite them using h, w, s and the symbols (logical operators).

(a) *John is smart or wealthy.*

(b) *John is healthy and wealthy.*

(c) *John is not wealthy.*

(d) *If John is healthy, then he is smart.*

Examples of compound statements

Solution

(a) $s \vee w$.

(b) $h \wedge w$.

(c) $\neg w$.

(d) $h \to s$.

Examples of compound statements

Solution

(a) $s \vee w$.

(b) $h \wedge w$.

(c) $\neg w$.

(d) $h \to s$.

Remark

When multiple operator appear, the priority order is

$$\neg > \wedge = \vee = \to$$

. *We need to add parentheses when necessary (ambiguity exists).*

Converse of statements

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Remark

In general, the converse of a statement is not equivalent to the original one. For example, "if n *is an integer, then* n^2 *is an integer" is true, while its converse is false.*

Contrapositive of statements

Definition

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Definition

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Remark

Both statements are false only when p *is true and* q *is false, so they are equivalent. In other words, the double implication*

$$(p \to q) \leftrightarrow (\neg q \to \neg p)$$

is always true.

Predicates

In practice, there are other statements that do not fit in the category of compound statements. We need the **predicate logic** too. In grammar, the word predicate refers to the part of a sentence that gives information about the subject.

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Definition

A **predicate** *is a sentence that contains a finite number of variables and becomes a statement when specific values are substituted for the variables. The domain of a predicate variable is the set of all values that may be substituted in place of the variable.*

Quantifiers

In a sentence, even if we fix the subject and the predicate, there is still a twist: the number of subjects referred to? For example, the following sentences have very different meanings.

- all humans are mortal;
- some humans are mortal;
- one human is mortal;
- no human is mortal.

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Definition

Quantifiers *are words that refer to quantities such as 'some' or 'all' and tell for how many subjects, a given predicate is true.*

The quantifiers $\forall$ and $\exists$

In predicate logic, there are two quantifiers that we use all the time: $\forall$ and $\exists$.

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Definition

The **universal quantifier**, *written as* $\forall$ *(LaTeX symbol \forall) and read 'for all', refers to all elements in the domain.*

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In predicate logic, there are two quantifiers that we use all the time: ∀ and ∃.

Definition

The **universal quantifier**, *written as* ∀ *(LaTeX symbol \forall) and read 'for all', refers to all elements in the domain.*

Definition

The **existential quantifier**, *written as* ∃ *(LaTeX symbol \exists) and read 'there exists/there exist', refers to at least one element in the domain.*

Negation of quantified statements

Let D be a domain and P be a predicate.

Proposition

The negation of

$$\forall x \in D, P(x)$$

is

$$\exists x \in D \text{ such that } \neg P(x).$$

Similarly, The negation of

$$\exists x \in D \text{ such that } P(x)$$

is

$$\forall x \in D, \neg P(x).$$

Example of quantified statements

Example

Find the negations of the following quantified statements:

(a) *There is a computer program written in the programming language* `Lisp`.

(b) *For any computer program, if it has more than 100,000 lines, then it contains a bug.*

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Example

Find the negations of the following quantified statements:

(a) *There is a computer program written in the programming language* `Lisp`.

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Solution

(a) There is no computer program written in the programming language `Lisp`.

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Find the negations of the following quantified statements:

(a) *There is a computer program written in the programming language* `Lisp`.

(b) *For any computer program, if it has more than 100,000 lines, then it contains a bug.*

Solution

(a) There is no computer program written in the programming language `Lisp`.
(b) There exists a computer program with more than 100,000 lines and it does not contain a bug.

Mathematical proofs

Proof techniques

There are several common techniques in mathematical proofs

- direct proof;
- proof by cases;
- proof by contrapositive;
- proof by contradiction.

Direct proof

Example

Prove that for any real number x, we have $x^2 + 1 > 0$.

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Proof.

Since x is a real number, its square is never negative, so x^2 is at least zero. Then $x^2 + 1 \geq 1 > 0$. $\square$

Proof by cases

Sometimes there are many cases in the problem, while in each case, since we have an additional assumption, it's easier to prove the statement.

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Prove that for any integer n, $n(n+1)$ *is even.*

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Example

Prove that for any integer n, $n(n+1)$ is even.

Proof.

We know that n is either odd or even.
If n is odd, then there exists an integer k such that $n = 2k+1$. Hence $n(n+1) = (2k+1)(2k+2) = 2(k+1)(2k+1)$ is even. If n is even, then there exists an integer k such that $n = 2k$. Hence $n(n+1) = 2k(2k+1)$ is even, too. $\square$

Proof by contrapositive

Since the contrapositive of a statement is equivalent to the original one, this provides an alternative way to prove a statement.

Example

If there are 85 *students in this class and there are* 3 *studio sessions, prove that there exists a session with at least* 29 *students.*

Proof by contrapositive

Since the contrapositive of a statement is equivalent to the original one, this provides an alternative way to prove a statement.

Example

If there are 85 students in this class and there are 3 studio sessions, prove that there exists a session with at least 29 students.

Proof.

We prove by contrapositive. Here p is "85 students belong to 3 sessions" and q is "there exists a session with at least 29 students". Suppose $\neg q$ is true, then each session has at most 28 students. So there are at most $28 \cdot 3 = 84$ students in all sessions, which is $\neg p$. So we proved $\neg q \to \neg p$. $\square$

Proof by contradiction

Sometimes a direct proof of a statement A seems hopeless. But we can make progress by assuming the negation of A. If we can then obtain a statement that is false or contradicts some true statement, we know that $\neg A$ is false and thus A is true.

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Prove that there is no largest integer.

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Example

Prove that there is no largest integer.

Proof.

We prove by contradiction. Suppose N is the largest integer. Then $N + 1$ is also an integer, and $N + 1 > N$, so N cannot be the largest integer and our assumption is false. So the original statement must be true. $\square$

Homework Assignment #1 - today's sections

Section 0.1 Exercise 2(f)-(n), 3(a)(b)(i)(j), 5(d)(h)(i)(j), 6(i)-(k), 7(a)(m)(n).
Section 0.2 Exercise 12, 22, 24, 25, 29.