Math 2603 - Lecture 10
Section 5.2 & 5.3 Recursively defined sequences

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September 19th, 2019

Problem-solving strategy: guess the answer, prove by induction

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Example

Let n be a positive integer. For positive integer k, $k!$ is the factorial of k, which is $\prod_{i=1}^{k} i$. Evaluate the following sum:

$$\sum_{k=1}^{n} (k \cdot k!).$$

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Hint

The first few terms in the sequence: $1, 4, 18, 96$. And the corresponding sums for small n: $1, 5, 23, 119$. Any pattern?

Problem-solving strategy: guess the answer, prove by induction

Solution

We claim that the sum is $(n+1)!-1$, *and we use induction to prove it. When* $n=1$, *the sum is* 1 *and* $(1+1)!-1=2-1=1$. *As for the inductive step, suppose* m *is an arbitrary positive integer such that* $\sum_{k=1}^{m} k\cdot k! = (m+1)!-1$. *Then*

$$\sum_{k=1}^{m+1} k\cdot k! = \sum_{k=1}^{m} k\cdot k! + (m+1)\cdot(m+1)!$$
$$=[(m+1)!-1]+(m+1)\cdot(m+1)!$$
$$=(m+2)\cdot(m+1)!-1=(m+2)!-1.$$

So the claim is still true when $n=m+1$. *We are done.*

Example: a flawed proof using induction

Example

Here is a "proof" of the false statement "for all integers $n \geq 1$, $3^n - 2$ *is even."*

Proof.

Suppose the statement is true for an arbitrary integer $k \geq 1$. *Then* $3^k - 2$ *is even. We must show that* $3^{k+1} - 2$ *is even. But*

$$3^{k+1} - 2 = 3^k \cdot 3 - 2 = (3^k - 2) + 2 \cdot 3^k.$$

Now $3^k - 2$ *is even by inductive hypothesis and* $2 \cdot 3^k$ *is even by definition. Hence their sum is also even. It follows that* $3^{k+1} - 2$ *is even, which is what we needed to show.* □

What is the flaw?

Induction makes no sense without basis step

Solution

All steps in the "proof" are correct, but it misses the basis step and in fact the basis step is apparently false.

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Remark

Although the inductive step is usually the essential step in a proof by induction, please note that it is only an implication! So if the premise is false, it is an invalid argument and it is useless. As a result, it is vital to make sure that the basis step is done correctly.

Example: a hidden flaw

Example

Here is a "proof" of the false statement "for all nonzero real numbers r and nonnegative integer n, $r^n = 1$."

Proof.

Fix r, we use strong induction on n. Basis step: when $n = 0$, since $r \neq 0$, $r^0 = 1$ is true.
Inductive step: suppose $k \geq 0$ is an arbitrary integer such that $r^i = 1$ for all $0 \leq i \leq k$. Note that $r^{k+1} = r^{k+k-(k-1)} = r^k \cdot r^k / r^{k-1}$. By the inductive hypothesis, $r^k = r^{k-1} = 1$, so r^{k+1} is also 1. The inductive step is done. □

What is the flaw?

Mind the range of numbers that the hypothesis applies to

Solution

The basis step is absolutely correct. In the inductive step, the formulas are correct. The flaw is that when $k = 0$, $k - 1 = -1$, which is no longer between 0 and k! So in this particular case we don't have $r^{k-1} = 1$!

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Solution

The basis step is absolutely correct. In the inductive step, the formulas are correct. The flaw is that when $k = 0$, $k - 1 = -1$, *which is no longer between* 0 *and* k*! So in this particular case we don't have* $r^{k-1} = 1$*!*

Remark

In this flawed proof, we can see that if we already have that the claim is true for $k = 0, 1$, *then the proof works. But this is expected - when* $k = 1$, *the claim becomes* $r = 1$, *and if* $r = 1$, *the statement would be true. Otherwise, it's false.*

Recursively defined sequences

Sequences

Definition

A **sequence** $\{a_n\}$ *is a function* f *whose domain is an infinite set of integers (often* $\mathbb{N}$*) and whose range is a subset of* $\mathbb{R}$*. For integer* m *in the domain, we usually write* a_m *for the value of* $f(m)$*.*

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Example

$f : \mathbb{N} \to \mathbb{R}$ *with* $f(n) = n^2$ *is the sequence* $1, 4, 9, 16, \cdots$*.*

Recurrence relation

Definition

A **recurrence relation** *for a sequence* $a_0, a_1, a_2, \cdots$ *is a formula that relates each term* a_k *to certain of its predecessors* $a_{k-1}, a_{k-2}, ..., a_{k-i}$, *where* i *is an integer such that* $k - i \geq 0$. *The initial conditions for such a recurrence relation specify the values of* $a_0, a_1, a_2, \cdots, a_{m-1}$, *where* m *is* i *or some other positive integer. The sequence* $\{a_n\}$ *is also called* **recursively defined**.

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Remark

i *is usually* 1 *or* 2*. The way we define a recurrence relation is very similar to the strong form of mathematical induction.*

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Remark

Given a recursively defined sequence, it may not be easy to find a direct formula for a_n *with general* n.

Some special sequences

Examples of special sequences

- Arithmetic progression;
- Geometric progression;
- Fibonacci sequence.

Arithmetic progressions

Definition

An **arithmetic progression** *is a sequence in which the difference of any two consecutive terms is a constant. In other words, it is a sequence of the form* $a, a+d, a+2d, \ldots$*, where* $a \in \mathbb{R}$ *is called the* **initial term** *and* $d \in \mathbb{R}$ *is called the* **common difference**.

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Remark

Alternatively, it is recursively defined as $a_{n+1} = a_n + d$. *The common difference* d *may be zero.*

Example: sum of terms in an arithmetic progression

Example

(1) Evaluate $\sum_{k=1}^{100} k$.
(2) Show that $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$ *for all positive integers* n.

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(2) Show that $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$ *for all positive integers* n.

Solution

(1) We can pair up integers from 1 *to* 100*:*
$1 + 100 = 101, 2 + 99 = 101, \cdots$. *There are* $100/2 = 50$ *pairs in total, and the sum of each pair is* 101*, so the total sum is* $101 \cdot 50 = 5050$.

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Remark

As a prodigy, German mathematician Carl Friedrich Gauss (1777-1855) managed to apply this method when he was a young boy.

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Solution

(2) We use induction on n. When $n = 1$, the claim becomes $1 = \frac{1 \cdot 2}{2}$, which is trivially true.

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Solution

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$$\sum_{k=1}^{m+1} k = \sum_{k=1}^{m} k + (m+1)$$

$$= \frac{m(m+1)}{2} + (m+1) = (m+1)(\frac{m}{2} + 1) = \frac{(m+1)(m+2)}{2}.$$

So the claim is still true when $n = m + 1$.

Sum of terms in general arithmetic progressions

Theorem

For positive integer n and real number d, we have:

$$\sum_{k=0}^{n-1}(a+kd) = na + \frac{n(n-1)}{2}d.$$

We can prove this theorem in both ways: pairs or induction.

Geometric progressions

Definition

An **geometric progression** *is a sequence in which the ratio of any two consecutive terms is a constant. In other words, it is a sequence of the form* $a, ar, ar^2, \ldots$*, where* $a \in \mathbb{R} - \{0\}$ *is called the* **initial term** *and* $r \in \mathbb{R} - \{0\}$ *is called the* **common ratio**.

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Remark

Be careful that the common ratio cannot be zero! As a corollary, all terms in a geometric progression may not be zero.

Sum of terms in general geometric progressions

Theorem

For positive integer n and real number $r \neq 0$, we have:

$$\sum_{k=0}^{n-1} ar^k = \begin{cases} na, & \text{if } r = 1; \\ a\frac{r^n - 1}{r-1}, & \text{if } r \neq 1. \end{cases}$$

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$$\sum_{k=0}^{n-1} ar^k = \begin{cases} na, & \text{if } r = 1; \\ a\frac{r^n - 1}{r - 1}, & \text{if } r \neq 1. \end{cases}$$

Proof.

The case when $r = 1$ is simple: $ar^k = a$ for all k, so the sum equals to na. Now we assume that $r \neq 1$. Let S be the sum. Then $rS = \sum_{k=0}^{n-1} ar^{k+1} = \sum_{k=1}^{n} ar^k$. So

$$rS - S = ar^n - a,$$

$$S = a\frac{r^n - 1}{r - 1}.$$

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Evaluate the following sum of terms in a geometric sequence:

$$\sum_{k=0}^{8} 2^k.$$

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Solution

Here we have a geometric sequence with 9 *terms. The initial term is* $2^0 = 1$ *and the common ratio is* 2*. By the theorem, the answer is*

$$a\frac{r^n - 1}{r - 1} = 1 \cdot \frac{2^9 - 1}{2 - 1} = 512 - 1 = 511.$$

Fibonacci sequence

Leonardo Fibonacci (1180-1228) was a prominent Italian mathematician. He considered the rapid reproduction of rabbit species and introduced the following sequence:

Definition

The Fibonacci sequence F_n *is defined as follows:*

- $F_0 = 0, F_1 = 1$;
- $F_n = F_{n-1} + F_{n-2}$ *for all integers* $n \geq 2$.

The terms F_n *are called* **Fibonacci numbers**.

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Remark

Fibonacci numbers have a lot of properties, and itself even became a small branch of mathematical research (there is even a research journal Fibonacci Quarterly).

How to find F_n

Remark

The first few terms of $\{F_n\}$ *are*

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \cdots$$

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Remark

We can easily prove that $\{F_n\}$ *is increasing, but it is neither arithmetic nor geometric, so how to find a formula for* F_n*?*

The characteristic polynomial

Example: first-order recurrence relation

Definition

A recurrence relation is called **first order** *if* $i = 1$. *In other words, the term* a_k *only depends on the value* a_{k-1}.

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Example

Let $a_1 = 1$ *and* $a_{k+1} = 2a_k + 1$ *for* $k \in \mathbb{N}$*. Find the value of* a_n*.*

Solution to first-order: make it geometric!

Remark

The first few terms are $1, 3, 7, 15, 31, \cdots$

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The first few terms are $1, 3, 7, 15, 31, \cdots$ *Each term is approximately twice as the previous term. So our sequence seems not very far from a geometric progression with common ratio* 2.

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Hint

It would be prefect if we don't have the 1 *in the recurrence relation. What if we shift all terms by the same number* c*?*

$$a_{k+1} + c = 2(a_k + c).$$

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$$a_{k+1} + c = 2(a_k + c).$$

If this holds, then $a_{k+1} = 2(a_k + c) - c = 2a_k + c$, *so* c *must be* 1.

Solution to example

Solution

Let $b_n = a_n + 1$ *for all* $n \in \mathbb{N}$. *Then* $b_1 = 2$ *and* $b_{k+1} = 2b_k$ *for all* $k \in \mathbb{N}$.

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$$a_n = b_n - 1 = 2^n - 1.$$

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Remark

For first-order recurrence relations, we usually modify the sequence such that the new sequence is geometric and we can easily find. Then we recover the original sequence. This is a special case of the general method - characteristic polynomial.

Solving Fibonacci sequence

Remark

Note that in $\{F_n\}$, *the value of each term depends on previous* 2 *terms. So it is of* **second-order**. *Is it still possible that* $\{F_n\}$ *is very similar to a geometric progression?*

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Hint

Suppose there is $a, r \neq 0$ *such that* $F_n = a \cdot r^n$*, then what property should* a *and* r *satisfy?*

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Suppose there is $a, r \neq 0$ *such that* $F_n = a \cdot r^n$*, then what property should* a *and* r *satisfy?*

$$a \cdot r^{n+2} = a \cdot r^{n+1} + a \cdot r^n.$$

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Hint

Suppose there is $a, r \neq 0$ such that $F_n = a \cdot r^n$, then what property should a and r satisfy?

$$a \cdot r^{n+2} = a \cdot r^{n+1} + a \cdot r^n.$$

Divided by nonzero $a \cdot r^n$, we get

$$r^2 = r + 1.$$

Solving Fibonacci sequence

Now we know that if a real number r satisfies $r^2 = r + 1$, then $F_n = a \cdot r^n$ satisfies the recurrence relation, all we need to fix is the values of the initial terms.

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The quadratic equation $x^2 - x - 1 = 0$ has 2 real roots $\frac{1 \pm \sqrt{5}}{2}$, and it turns out that the Fibonacci numbers are a linear combination of these two:

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The quadratic equation $x^2 - x - 1 = 0$ has 2 real roots $\frac{1 \pm \sqrt{5}}{2}$, and it turns out that the Fibonacci numbers are a linear combination of these two:

Theorem

$$F_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1 + \sqrt{5}}{2} \right)^n - \left(\frac{1 - \sqrt{5}}{2} \right)^n \right] \forall n \in \mathbb{Z}, n \geq 0.$$

The characteristic polynomial

Definition

If the recurrence relation is $a_{k+2} = ra_{k+1} + sa_k$ *where* $r, s \in \mathbb{R}$, *its* **characteristic polynomial** *is* $x^2 - rx - s$.

The characteristic polynomial

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Theorem

Let x_1 *and* x_2 *be the roots of the polynomial* $x^2 - rx - s$*. Then the solution of the recurrence relation* $a_n = ra_{n-1} + sa_{n-2}, n \geq 2$ *is*

$$a_n = \begin{cases} c_1 x_1^n + c_2 x_2^n, & \text{if } x_1 \neq x_2; \\ c_1 x^n + c_2 n x^n, & \text{if } x_1 = x_2 = x. \end{cases}$$

Example: second-order recurrence relation

Example

Solve the recurrence relation $a_n = 5a_{n-1} - 6a_{n-2}$ *for* $n \geq 2$ *and* $a_0 = -3, a_1 = -2$.

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Hint

By the theorem above, we solve the characteristic polynomial and know the general form of solutions, with constants c_1, c_2 *to be determined.*

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Hint

By the theorem above, we solve the characteristic polynomial and know the general form of solutions, with constants c_1, c_2 *to be determined. The two initial terms would establish two linear equations for* c_1, c_2*, which leads to a unique solution. For more details, see Section 5.3 Exercise 25 in Homework.*

Example: second-order recurrence relation

Solution

Here $r = 5, s = -6$. *The characteristic polynomial is* $x^2 - 5x + 6 = (x - 2)(x - 3)$. *So the roots are* 2 *and* 3.

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Note that this is also true when $n = 0, 1$. *We then plug-in* $n = 0, 1$*:*

$$\begin{aligned} -3 = a_0 &= c_1 2^0 + c_2 3^0 = c_1 + c_2 \\ -2 = a_1 &= c_1 2^1 + c_2 3^1 = 2c_1 + 3c_2. \end{aligned}$$

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$$-3 = a_0 = c_1 2^0 + c_2 3^0 = c_1 + c_2$$
$$-2 = a_1 = c_1 2^1 + c_2 3^1 = 2c_1 + 3c_2.$$

Finally, we solve for c_1, c_2.
$c_2 = (2c_1 + 3c_2) - 2(c_1 + c_2) = -2 - 2 \cdot (-3) = 4$. *And* $c_1 = -3 - c_2 = -7$. *The solution is* $a_n = (-7) \cdot 2^n + 4 \cdot 3^n$.

HW Assignment #5 - today's sections

Section 5.2 Exercise 5, 6, 33(c)(d), 48, 51.
Section 5.3 Exercise 2, 11, 25.