Math 2603 - Lecture 13
Section 6.1 & 6.2 Principles of counting

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Addition and Multiplication Rules

Example: counting a process with multiple steps

Example

Suppose during a weekend, Atlanta's football team, baseball team and soccer team each play a regular season game. Both football and baseball games must have a winner, while the soccer game could be a draw. How many possible outcomes for the series of these 3 *games?*

Example: counting a process with multiple steps

Solution

Note that there are 2 possible outcomes for football and baseball and 3 for soccer. We have the following table of all possible outcomes.

Football	*W*	*W*	*W*	*W*	*W*	*W*	*L*	*L*	*L*	*L*	*L*	*L*
Baseball	*W*	*W*	*W*	*L*	*L*	*L*	*W*	*W*	*W*	*L*	*L*	*L*
Soccer	*W*	*D*	*L*	*W*	*D*	*L*	*W*	*D*	*L*	*W*	*D*	*L*

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Soccer	*W*	*D*	*L*	*W*	*D*	*L*	*W*	*D*	*L*	*W*	*D*	*L*

The answer is 12, which equals $2 \cdot 2 \cdot 3$.

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The answer is 12, which equals $2 \cdot 2 \cdot 3$.

Remark

Note that we get an outcome of the football game first, then independently we get another result of the baseball game, and finally the soccer game. In general, there is a pattern for such situations with independent steps.

The Multiplication Rule

Definition

If an operation consists of k steps and

- *the first step can be performed in n_1 ways,*
- *the second step can be performed in n_2 ways (regardless of how the first step was performed),*
- $\cdots$
- *the k-th step can be performed in n_k ways (regardless of how the preceding steps were performed),*

then the entire operation can be performed in

$$\prod_{i=1}^{k} n_i = n_1 n_2 \cdots n_k$$

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Exercise: two letter postal codes

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USPS uses a two-letter code for every US state (and territory). For example, the code for the state of Georgia is GA. Suppose both letters in such codes could be any letter in the English alphabet. How many possible two-letter postal codes are there in total? (Of course many of them are not actually in use)

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We apply the multiplication rule. In order to get a two-letter code, we need to fix the first letter and then fix the second letter.

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Solution

We apply the multiplication rule. In order to get a two-letter code, we need to fix the first letter and then fix the second letter. The first letter could be any of the 26 *letters, so there are* $n_1 = 26$ *ways for the first step. Similarly, there are* $n_2 = 26$ *steps for the second step. So the answer is just* $26 \cdot 26 = 676$.

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In order to count, we need to determine the number of letters in the password. So we are doing a "solution by cases". If the password has 1 *letter, there are* 26 *such passwords. If the password has* 2 *letters, by the multiplication rule, there are* $26^2 = 676$ *such passwords. If the password has* 3 *letters, by the multiplication rule, there are* 26^3 *such passwords.*

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In summary, the answer is $26 + 26^2 + 26^3 = 18,278$.

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Theorem

Suppose a finite set A equals the union of k distinct **mutually disjoint** *subsets $A_1, A_2, \ldots, A_k$. Then*

$$|A| = |A_1| + |A_2| + \cdots + |A_k|.$$

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In other words, if an operation consists of one step which could be done in k mutually exclusive cases, then the total number of ways is the sum of the number of ways in those k cases.

Example: combination of the two rules

Example

Suppose you order lunch at Panda Express. For proteins, you may choose from rice, chow mein, vegetables or nothing; for entrees, you may choose 1 *or* 2 *entrees from* 10 *choices. How many different lunch boxes could you have?*

Solution

First, your choice consists of two steps:

- *choice of proteins;*
- *choice of entrees.*

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So in the end we apply multiplication rule. While within each step, we might apply addition rule. (to be continued)

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For proteins, you may choose 1 *from* 3, *or choose nothing, so there are* $3+1=4$ *choices of proteins;*

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Remark

For general counting problems, we usually combine the two rules - first divide the task into independent steps, then within each step we apply addition rule to count the number of choices of each step, in the end we apply multiplication rule to get the answer.

The difference rule

Theorem

If A is a finite set and B is a subset of A, then

$$|A \backslash B| = |A| - |B|.$$

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Remark

The difference rule provides an alternative way for counting: one can count the cardinality of a bigger set first, then subtract the number of excessive elements.

Principle of Inclusion-Exclusion

Dealing with repeated elements

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When you roll two dice, how many ways can you get at least one 6*?*

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We apply addition rule. There are two cases: the first die is 6 *or the second die is* 6*.*

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Solution

We apply addition rule. There are two cases: the first die is 6 *or the second die is* 6*. if the first die is* 6*, the second die could be anything, so there are* 6 *ways; if the second die is* 6*, the first die could be anything, so there are* 6 *ways; The answer is* $6 + 6 = 12$*. Are we done?*

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Are we done?
No! The case when both dice are 6 *is counted twice! The correct answer is* $6 + 6 - 1 = 11$*.*

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When you roll two dice, how many ways can you get at least one 6?

Solution

We apply addition rule. There are two cases: the first die is 6 or the second die is 6. if the first die is 6, the second die could be anything, so there are 6 ways; if the second die is 6, the first die could be anything, so there are 6 ways; The answer is $6 + 6 = 12$.
Are we done?
No! The case when both dice are 6 is counted twice! The correct answer is $6 + 6 - 1 = 11$.

Remark

Note that in order to apply the addition rule, the subsets must be mutually disjoint. What about other cases?

The Inclusion-Exclusion principle for two and three sets

Theorem

If A, B, C are any finite sets, then

$$|A \cup B| = |A| + |B| - |A \cap B|$$

and

$$\begin{aligned} |A \cup B \cup C| = |A| &+ |B| + |C| \\ &- |A \cap B| - |A \cap C| - |B \cap C| \\ &+ |A \cap B \cap C|. \end{aligned}$$

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Remark

There is a general formula for n sets, while we usually use the above formulas in most cases.

The general Inclusion-Exclusion

Theorem

Let $n \in \mathbb{N}$. $A_1, A_2, \cdots, A_n$ *are finite sets. Then we have*

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} (-1)^{i+1} \cdot \sum_{1 \le j_1 < \ldots < j_i \le n} \left|\bigcap_{k=1}^{i} A_{j_k}\right|.$$

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Note that the signs of terms on right hand side are alternating.

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Remark

Note that the signs of terms on right hand side are alternating. We can prove the general version by induction, or by check how many times each element is counted in both sides.

Exercise: multiples of 3 and 5

Example

How many integers from 1 *through* 100 *are multiples of* 3 *or multiples of* 5*? How many integers from* 1 *through* 100 *are neither multiples of* 3 *or multiples of* 5*?*

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Solution

Let m_k be the set of multiples of k from 1 through 100. We need to compute $|m_3 \cup m_5|$. By the inclusion/exclusion rule, it equals $|m_3| + |m_5| - |m_3 \cap m_5|$. Note that an integer is a multiple of both 3 and 5 if and only if it is a multiple 15. In addition, $|m_k| = \lfloor \frac{100}{k} \rfloor$, so

$$|m_3 \cup m_5| = \left\lfloor \frac{100}{3} \right\rfloor + \left\lfloor \frac{100}{5} \right\rfloor - \left\lfloor \frac{100}{15} \right\rfloor = 33 + 20 - 6 = 47.$$

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By the difference rule, the answer is (let $[100]$ be the universal set) $|\,(m_3 \cup m_5)^c\,| = 100 - |m_3 \cup m_5| = 100 - 47 = 53.$

Example: European countries

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Currently, European Union (EU) consists of 28 *countries. The Schengen Area consists of* 26 *countries. Iceland, Liechtenstein, Norway and Switzerland are the only* 4 *countries that belong to the Schengen Area but are not members of the EU. How many countries belong to at least one of the EU and the Schengen Area?*

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Solution

We know that $|EU| = 28$, $|Schengen| = 26$ *and* $|Schengen - EU| = 4$. *Since for any two sets* A, B, $A = (A \cap B) \cup (A - B)$, *and the two parts are disjoint. Then* $|Schengen \cap EU| = 26 - 4 = 22$. *And by the Inclusion-Exclusion principle,*

$$\begin{aligned}|Schengen \cup EU| = |EU| + |Schengen| \\ - |Schengen \cap EU| = 28 + 26 - 22 = 32.\end{aligned}$$

A summary of cardinality properties

Proposition

Let A and B be subsets of a finite universal set U. Then

1. $|A \cup B| = |A| + |B| - |A \cap B|$.
2. $|A \cap B| \leq \min(|A|, |B|)$.
3. $|A \backslash B| = |A| - |A \cap B| \geq |A| - |B|$.
4. $|A^c| = |U| - |A|$.
5. $|A \times B| = |A| \cdot |B|$.

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Remark

(1): Inclusion-Exclusion; (2) subset; (3); addition rule; (4) difference rule; (5) multiplication rule.

HW Assignment #7 - today's sections

Section 6.1 Exercise 4, 11, 21.

Section 6.2 Exercise 1, 4(b), 8, 12.