Math 2603 - Lecture 14
Section 6.3 & 7.1 Pigeonhole Principle and permutations

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The principle

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Theorem (The pigeonhole principle)

If n objects are put into m boxes and $n > m$, then at least one box must contain two or more objects.

Interpretation: contrapositive

Remark

Why this principle is true?

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Why this principle is true? Suppose we put objects into m *boxes. Then we have the following true statement of implication:*
if there is at most 1 *object in each box* (p)*, then there are at most* $1 \cdot m = m$ *objects in total* (q)*.*

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if there are more than m *objects in total, then it's not the case that there is at most* 1 *object in each box, in other words, there exists a box containing* 2 *or more objects.*

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if there are more than m *objects in total, then it's not the case that there is at most* 1 *object in each box, in other words, there exists a box containing* 2 *or more objects.*
This is exactly the pigeonhole principle. So in fact it's very straightforward to understand.

Exercise: birthday

Example

There are 15 *students coming to my office hour. Prove that there must be two students whose birthdays are in the same month of the year.*

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Proof.

Here the boxes are the 12 months, and the objects are the students.

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There are 15 students coming to my office hour. Prove that there must be two students whose birthdays are in the same month of the year.

Proof.

Here the boxes are the 12 months, and the objects are the students. There are 12 months of a year and 15 students, since $15 > 12$, by the pigeonhole principle, there must be two students whose birthdays are in the same month of the year. □

Example: relative prime

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Proof.

We construct 2 boxes: $\{n, n+1\}$ and $\{n+2, n+3\}$. There are 3 chosen integers, by the pigeonhole principle, there exists a box with at least 2 chosen numbers.

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We construct 2 boxes: $\{n, n+1\}$ and $\{n+2, n+3\}$. There are 3 chosen integers, by the pigeonhole principle, there exists a box with at least 2 chosen numbers. Since the box has exactly two elements, both must have been chosen, and they are relatively prime. □

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Remark

We can also prove by cases, if $n, n+1, n+2$ *are chosen, then the statement is true because* $\gcd(n, n+1) = 1$*; if* $n, n+2, n+3$ *are chosen, then the statement is true because* $\gcd(n+2, n+3) = 1$*, and so on.*

Constructive proofs vs existential proofs

For statements about the existence of certain objects, there is a dichotomy of proofs:

Definition

A **constructive proof** *justifies the existence by explicitly constructing or finding the desired objects. In comparison, an* **existential proof** *justifies their existence by axioms, true statements and logic, but without explicitly pointing out what they are.*

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Remark

Naturally, the pigeonhole principle is frequently used in existential proofs.

Alternative Forms

The strong form

In fact, when n is much larger than m, we can get a stronger conclusion:

Theorem (The strong form of pigeonhole principle)

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We prove by contrapositive. If all boxes have at most $\lceil \frac{n}{m} \rceil - 1$ objects, then there are at most $m \cdot \left(\lceil \frac{n}{m} \rceil - 1 \right)$ objects in total.

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Remark

$\left\lceil \frac{n}{m} \right\rceil$ *is the best number we can get.*

Example: studio sections

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Proof.

We apply the strong form. Let $n = 109$ and $m = 3$, then there exists a studio section with at least $\left\lceil \frac{109}{3} \right\rceil = 37$ students. □

Exercise: number of picks needed to ensure a result

Example

Suppose in an election there are 10 *candidates, and each voter votes for exactly one of them. What is the smallest number of voters such that no matter how they vote, there is always a candidate who gets at least* 5 *votes?*

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Solution

Suppose there are n voters, by the strong form of pigeonhole principle, there is always a candidate who gets at least $\left\lceil \frac{n}{10} \right\rceil$ votes. So if this number is 5, n is good.

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$$\left\lceil \frac{40}{10} \right\rceil = 4, \left\lceil \frac{41}{10} \right\rceil = 5.$$

So 41 voters is enough. While 40 voters may not work if each candidate gets exactly 4 votes. So the answer is 41.

Example: how to construct the boxes

Example

Prove that for any sequence of integers $a_1, a_2, \cdots, a_{10}$*, there is a string of consecutive integers among them* $a_l, a_{l+1}, \cdots, a_k$ *whose sum is a multiple of* 10.

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Proof.

Consider the numbers

$$a_1, a_1 + a_2, \cdots, a_1 + \ldots + a_{10}.$$

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Prove that for any sequence of integers $a_1, a_2, \cdots, a_{10}$, there is a string of consecutive integers among them $a_l, a_{l+1}, \cdots, a_k$ whose sum is a multiple of 10.

Proof.

Consider the numbers

$$a_1, a_1 + a_2, \cdots, a_1 + \ldots + a_{10}.$$

If any one of them is a multiple of 10, we are done. Otherwise they belong to the remaining 9 congruence classes modulo 10. By Pigeonhole principle, there exists two numbers that are congruent modulo 10. Hence their difference is a multiple of 10 and it is still the sum of consecutive integers among the sequence. □

The infinite and uncountable form

Theorem (Infinite version of pigeonhole principle)

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Theorem (Uncountable version of pigeonhole principle)

If uncountably many objects are put into countably many boxes, then at least one box must contain uncountably many objects.

Remark

Both proofs are simple in terms of contrapositive.

Example: cardinality of $[0, 1]$

Definition

For $x \in \mathbb{R}$, the **fractional part** *of x, denoted $\{x\}$, is defined as $x - \lfloor x \rfloor$. Note that $0 \leq \{x\} < 1$ for all $x \in \mathbb{R}$.*

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Given that $\mathbb{R}$ is uncountable, prove that $[0, 1]$ is uncountable too.

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Example

Given that $\mathbb{R}$ is uncountable, prove that $[0, 1]$ is uncountable too.

Proof.

We consider the function $x \mapsto \{x\}$ defined on $\mathbb{R}$. It's range is $[0, 1)$. Note that for each $a \in [0, 1)$, the preimage of a is $\{a + k \mid k \in \mathbb{Z}\}$, which is countable. If $[0, 1)$ is countable too, then $\mathbb{R}$ is the union of countably many preimages, so a union of countably many countable sets, then $\mathbb{R}$ is countable, a contradiction! Hence $[0, 1)$ is uncountable, so is $[0, 1]$. $\square$

Permutations

Example: positions for interns

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Suppose there is one internship position in each of the following three departments of a company: marketing, technology and management and the HR chief needs to select one student for each department. There are 5 *college students to select from. How many possible arrangements are there?*

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Solution

We apply the multiplication rule. We determine the selection of marketing, then technology, and finally management.

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Solution

We apply the multiplication rule. We determine the selection of marketing, then technology, and finally management. There are 5 *ways to select the intern for marketing, and then* 4 *ways for technology and* 3 *ways for management. So in total* $5 \cdot 4 \cdot 3 = 60$ *arrangements.*

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Proposition

For any integer $n \geq 1$, the number of permutations of a set with n elements is $n!$.

The proof is just an application of the multiplication rule.

Permutations of Selected Elements

A more general case is the permutations of selected elements.

Definition

An r-**permutation** *of a set of* n *elements is an ordered selection of* r *elements taken from the set of* n *elements. The number of* r*-permutations of a set of* n *elements is denoted* $P(n, r)$.

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Example

$P(n, 1) = n, P(n, n) = n!$ *for each positive integer n.*

Exercise: license plates

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Solution

First we apply the multiplication rule. We determine the letters first and then the digits. For the letters, we select the first one, which has 26 *choices. For the second letter, it could be anything but the first letter, so it has* $26 - 1 - 25$ *choices. And the third letter has* 24 *choices. Similarly, the four digits have* $10, 9, 8, 7$ *choices in total. So the answer is*

$$P(26,3) \cdot P(10,4) = 26 \cdot 25 \cdot 24 \cdot 10 \cdot 9 \cdot 8 \cdot 7 = 78,624,000.$$

Formula of $P(n, r)$

Theorem

$$P(n, r) = n(n-1)(n-2)\cdots(n-r+1) = \frac{n!}{(n-r)!}.$$

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Theorem

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Proof.

We apply the multiplication rule. There are n choices for the first element, then $n-1$ choices for the second element, and so on. As for the last element, it could be anything but the previously chosen $r-1$ elements, so it has $n-(r-1)$ choices. Then we have the above formula. □

Example: evaluate $P(n, r)$

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Evaluate $P(5, 3)$ *and* $P(6, 3)$.

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Solution

$P(5,3) = \frac{5!}{(5-3)!} = \frac{120}{2} = 60.$

Example: evaluate $P(n, r)$

Example

Evaluate $P(5,3)$ *and* $P(6,3)$.

Solution

$P(5,3) = \frac{5!}{(5-3)!} = \frac{120}{2} = 60.$ $P(6,3) = \frac{6!}{(6-3)!} = \frac{720}{6} = 120.$

Homework Assignment #8 - today

Section 6.3 Exercise 6, 9, 12, 30.

Section 7.1 Exercise 4, 9, 15.