Math 2603 - Lecture 15
Section 7.2 & 7.7 Combinations and the Binomial Theorem

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Combinations

Unordered selections

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Example

- *One can select 3 toppings from a list on a large pizza.*
- *The coach can select any 5 players from the team roster to start in a basketball game.*
- *In a sweepstakes, our local grocery store selects 10 customers to receive the same prize.*

r-combinations

Definition

Let n and r be nonnegative integers with $r \leq n$. An r-**combination** *of a set of n elements is a subset of r of the n elements. The symbol*

$$\binom{n}{r},$$

which is read "n choose r" (LaTeX symbol \binom{n}{r}), denotes the number of subsets of size r (r-combinations) that can be chosen from a set of n elements.

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Example

$$\binom{n}{1} = n, \binom{n}{n} = 1, \binom{n}{0} = 1.$$

A direct formula of $\binom{n}{r}$

Theorem

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There are $\frac{n!}{(n-r)!}$ r-permutations. For each r-combination, how many r-permutations does it correspond to?

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Proof.

There are $\frac{n!}{(n-r)!}$ r-permutations. For each r-combination, how many r-permutations does it correspond to? Since the r elements are fixed, they are just the r-permutations on r elements, which means the number is $r!$ So each r-combination appears in $r!$ r-permutations. And thus

$$\binom{n}{r} = \frac{n!}{(n-r)!}/r! = \frac{n!}{r!(n-r)!}.$$

$\square$

Example: Powerball lottery

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One buys a Powerball lottery ticket as follows: select five distinct unordered numbers from 1 *to* 69 *for the white balls, then select one number from* 1 *to* 26 *for the red Powerball. There is only one winning number of* 5 *white balls and* 1 *red ball for the top prize. How many possible outcomes of the winning numbers?*

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Solution

To get a combination, there are two steps: choose five white balls and one red ball. So we apply the multiplication rule first. For the white balls, they are a 5 *selection among* 69 *numbers, so the number of ways is* $\binom{69}{5}$*;*

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Solution

To get a combination, there are two steps: choose five white balls and one red ball. So we apply the multiplication rule first. For the white balls, they are a 5 selection among 69 numbers, so the number of ways is $\binom{69}{5}$; for the red ball, it is a single choice from the 26 numbers, so there are 26 choices. The answer is

$$\binom{69}{5} \cdot 26 = 292,201,338.$$

Example: executive board

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Solution

We break it into independent steps:

- *president, secretary and treasurer -* $P(30,3) = 30 \cdot 29 \cdot 28$;
- *two other officers -* $\binom{27}{2}$.

So the answer is $P(30,3) \cdot \binom{27}{2} = \frac{30 \cdot 29 \cdot 28 \cdot 27 \cdot 26}{2} = 8,550,360$.

The Binomial Theorem

Properties of $\binom{n}{r}$

Proposition

For nonnegative integers $n \geq r$, we have

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Fix n. The maximum of $\binom{n}{r}$ is obtained at which integer r?

Proposition

For any positive integer n *and integer* $0 \leq r \leq n$, *we have*

$$\binom{n}{r} \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

Pascal's formula

A natural question about $\binom{n}{r}$ is: how to compute it? We know that factorials work. However, it is a very inefficient method because many factors will be canceled out in the end and to compute the products in the factorials seems unnecessary.

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Theorem (Pascal's formula)

For nonnegative integers n, r with $n+1 \geq r$, we have

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For nonnegative integers n, r with $n+1 \geq r$, we have

$$\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}.$$

Remark

This is a recursive relation because all $\binom{n}{\cdot}$ values give all $\binom{n+1}{\cdot}$.

The proof of Pascal's formula

Proof.

$$\begin{aligned}\binom{n}{r-1}+\binom{n}{r} &= \frac{n!}{(r-1)!(n-r+1)!}+\frac{n!}{r!(n-r)!}\\ &= \frac{n!\cdot r}{r!(n-r+1)!}+\frac{n!\cdot(n-r+1)}{r!(n-r+1)!}\\ &= \frac{n!\cdot[r+(n-r+1)]}{r!(n-r+1)!}=\frac{n!\cdot(n+1)}{r!(n-r+1)!}\\ &= \frac{(n+1)!}{r!(n-r+1)!}=\binom{n+1}{r}.\end{aligned}$$

□

A proof by counting

There is an alternative proof by counting.

Proof.

$\binom{n+1}{r}$ is the number of r-combination among integers 1 through $n+1$. Then there are two disjoint cases: 1 is in the combination or not.

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Pascal's triangle

The **Pascal's triangle** is a triangular array of numbers such that the n-th row are just the numbers

$$\binom{n-1}{0}, \binom{n-1}{1}, \binom{n-1}{2}, \dots, \binom{n-1}{n-1}.$$

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$$\begin{array}{ccccccccccccccc}
&&&&&&&1&&&&&&& \\
&&&&&&1&&1&&&&&& \\
&&&&&1&&2&&1&&&&& \\
&&&&1&&3&&3&&1&&&& \\
&&&1&&4&&6&&4&&1&&& \\
&&1&&5&&10&&10&&5&&1&& \\
&1&&6&&15&&20&&15&&6&&1& \\
1&&7&&21&&35&&35&&21&&7&&1
\end{array}$$

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$$\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & & & & 1 & & 1 & & & & & & \\
 & & & & & 1 & & 2 & & 1 & & & & & \\
 & & & & 1 & & 3 & & 3 & & 1 & & & & \\
 & & & 1 & & 4 & & 6 & & 4 & & 1 & & & \\
 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & \\
 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & \\
1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1
\end{array}$$

Remark

French mathematician Blaise Pascal (1623-1662) studied it in the 17*th century, while scholar from India, Iran, China and Italy also studied it earlier.*

Example: next row in the Pascal's triangle

Example

The 8-th row of the Pascal's triangle has numbers

$$1, 7, 21, 35, 35, 21, 7, 1.$$

Find the numbers in the 9-th row of the Pascal's triangle.

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$$1, 7, 21, 35, 35, 21, 7, 1.$$

Find the numbers in the 9-th row of the Pascal's triangle.

Solution

The numbers are

$$1, 1+7, 7+21, 21+35, 35+35, \cdots$$

which are

$$1, 8, 28, 56, 70, 56, 28, 8, 1.$$

Binomials

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In algebra, a sum of two terms, such as $a + b$*, is called a* **binomial**.

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Remark

Except for the monomials, binomials are the simplest expressions, so we want to study their arithmetic operations, especially their product.

Example: expand a power of binomial

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Expand the product $(a+b)^4$.

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Solution

$$\begin{aligned}
(a+b)^4 &= \left[(a+b)^2\right]^2 \\
&= \left[a^2+2ab+b^2\right]^2 \\
&= \left(a^4+2a^3b+a^2b^2\right)+\left(2a^3b+4a^2b^2+2ab^3\right) \\
&+\left(a^2b^2+2ab^3+b^4\right) \\
&= a^4+4a^3b+6a^2b^2+4ab^3+b^4.
\end{aligned}$$

The binomial theorem

The coefficients in the expansion of $(a+b)^4$ are the numbers "n choose r". This is not a coincidence. In fact, $\binom{n}{r}$ are called **binomial coefficients**.

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For any real numbers a, b and nonnegative integer n,

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}.$$

Proof.

Each term in the expansion is of the form $a^k b^{n-k}$. Fix k. To get such a term we need exactly k copies of a and $n-k$ copies of b, which means we need to choose k parentheses for a among the n parentheses. So this number is $\binom{n}{k}$. $\square$

Example: applications of the binomial theorem

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Show that for every positive integer n*,*

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Proof.

We plug-in $a = b = 1$ in the binomial theorem, the left hand side becomes $(1+1)^n = 2^n$ and the right hand side becomes $\sum_{k=0}^{n} \binom{n}{k} \cdot 1^k \cdot 1^{n-k} = \sum_{k=0}^{n} \binom{n}{k}$. $\square$

Homework Assignment #8 - today

Section 7.2 Exercise 8, 10, 17(a)(d), 23.
Section 7.7 Exercise 6, 10, 19, 21.