Math 2603 - Lecture 16
Section 7.3 & 7.4 Probability

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October 17th, 2019

About midsemester survey

Main concerns

- Quizzes are too hard
- Quizzes are not very related to HW and lecture
- Examples in lecture are too few and too easy
- Office hour time is bad
- Homework solutions contain error

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Remark

I promise that I will definitely curve your quiz grades in the end, to make sure that the average is about 80%.

Main Praises

- Lecture slides are helpful
- Answers on Piazza are clear
- Midterm #1 is fair

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Remark

I will continue on these efforts.

Elementary Probability

Motivating Examples

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- *The probability that an odd number appears when a fair die is rolled is* $\frac{3}{6} = \frac{1}{2}$.

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Solution

The number of possible outcomes of the 8 tosses is $2^8 = 256$.
The number of possible outcomes with 5 heads is the number of 5-combinations of 8 elements, which is

$$\binom{8}{5} = \binom{8}{3} = \frac{8 \cdot 7 \cdot 6}{3 \cdot 2 \cdot 1} = \frac{8 \cdot 7 \cdot 6}{6} = 56.$$

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Solution

The number of possible outcomes of the 8 tosses is $2^8 = 256$.
The number of possible outcomes with 5 heads is the number of 5-combinations of 8 elements, which is

$$\binom{8}{5} = \binom{8}{3} = \frac{8 \cdot 7 \cdot 6}{3 \cdot 2 \cdot 1} = \frac{8 \cdot 7 \cdot 6}{6} = 56.$$

So the probability is $\frac{56}{256} = \frac{7}{32}$.

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Remark

They are all related to gambling, which was the motivation of probability theory back in $17th$ *century.*

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They are all related to gambling, which was the motivation of probability theory back in 17th century.

Remark

In these examples, all possible outcomes are **equally likely** *to happen. This property enables a simple computation of the probability.*

The definition

Definition

A set S *of possible outcomes is called the* **sample space** *of an* **experiment**. *An* **event** *is a subset* A *of the sample space. The* **probability** *of event* A*, denoted as* $P(A)$*, measures how likely the event* A *will happen. More precisely, how likely any possible outcome in the subset* A *will happen.*

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Definition

If all possible outcomes in a finite sample space S are equally likely to happen, then for any event $A \subset S$, we have

$$P(A) = \frac{|A|}{|S|}.$$

The punchline

Remark

The previous formula is the key in probability theory. With it, we simply need to do the following 4 steps to compute probabilities:

1. *Find sample space S with equally likely possible outcomes.*
2. *Count the cardinality $|S|$ of S.*
3. *For a given event $A \subset S$, count the cardinality $|A|$ of A.*
4. *Compute $\frac{|A|}{|S|}$.*

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Usually it's easy to find S and compute $|S|$, while sometimes you need to make sure that the possible outcomes are indeed equally likely.

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Remark

Usually it's easy to find S and compute $|S|$, while sometimes you need to make sure that the possible outcomes are indeed equally likely. Then the essential step is to find $|A|$. **Generally speaking, elementary probability problems are just counting problems.**

More examples

Example

A committee of 5 *people is randomly chosen from* 4 *men and* 6 *women. Find the probability of the following events:*

1. *Exactly* 4 *women are on the committee.*
2. *At least* 4 *women are on the committee.*

More examples

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A committee of 5 *people is randomly chosen from* 4 *men and* 6 *women. Find the probability of the following events:*

1. *Exactly* 4 *women are on the committee.*
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Solution

S *consists of all* 5*-combinations of the* $4 + 6 = 10$ *people, so* $|S| = \binom{10}{5} = 252$. *(to be continued)*

More examples

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Solution

S consists of all 5-combinations of the $4+6=10$ people, so $|S| = \binom{10}{5} = 252$. (1) Let A be the event. It suffices to count $|A|$. If there are exactly 4 women on the committee, then the remaining 1 member must be a man. So $|A| = \binom{6}{4} \cdot \binom{4}{1} = 15 \cdot 4 = 60$.

More examples

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Solution

S consists of all 5-combinations of the $4+6=10$ people, so $|S| = \binom{10}{5} = 252$. (1) Let A be the event. It suffices to count $|A|$. If there are exactly 4 women on the committee, then the remaining 1 member must be a man. So $|A| = \binom{6}{4} \cdot \binom{4}{1} = 15 \cdot 4 = 60$. And the probability is $\frac{|A|}{|S|} = \frac{60}{252} = \frac{5}{21}$.

(2) Let B be the event. There are two cases: 4 women are on the committee or 5 women are on the committee. By the addition rule, $|B|$ is the sum of possible outcomes in both cases, which is

$$\binom{6}{4}\binom{4}{1} + \binom{6}{5}\binom{4}{0} = 60 + 6 = 66.$$

Hence the probability is $\frac{|B|}{|S|} = \frac{66}{252} = \frac{11}{42}$.

More examples

Example

A box contains 30 *tickets, each labeled with distinct integers from* 1 *to* 30 *inclusive. Find the probability that a ticket drawn randomly from the box bears the number that is divisible by* 3 *or* 5*.*

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Solution

The sample space S *has cardinality* 30. *The event* A *is* $\{x \in \mathbb{N} \mid x \leq 30, 3 \mid x \text{ or } 5 \mid x\}$.

More examples

Example

A box contains 30 tickets, each labeled with distinct integers from 1 to 30 inclusive. Find the probability that a ticket drawn randomly from the box bears the number that is divisible by 3 or 5.

Solution

The sample space S has cardinality 30. The event A is $\{x \in \mathbb{N} \mid x \leq 30, 3 \mid x$ or $5 \mid x\}$. By the Principle of Inclusion-Exclusion,

$$|A| = \left\lfloor \frac{30}{3} \right\rfloor + \left\lfloor \frac{30}{5} \right\rfloor - \left\lfloor \frac{30}{3 \cdot 5} \right\rfloor = 10 + 6 - 2 = 14.$$

Hence the probability is $\frac{|A|}{|S|} = \frac{14}{30} = \frac{7}{15}$.

Properties of probability

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Let S be the finite sample space of some experiment.

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3. *If A and B are events, then $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.*

Corollary of Principle of Inclusion-Exclusion

Theorem

Let S be the finite sample space of some experiment and $A_1, A_2, \cdots, A_n$ be events, then

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n}\left[(-1)^{i+1} \cdot \sum_{1 \le j_1 < \ldots < j_i \le n} P\left(\bigcap_{k=1}^{i} A_{j_k}\right)\right].$$

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Corollary

Let S be the finite sample space of some experiment and $A_1, A_2, \cdots, A_n$ be pairwise mutually exclusive events. Then

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i).$$

Probability Theory

Necessary condition of the function $P(A)$

Remark

Suppose S is a sample space containing possible outcomes $x_1, \ldots, x_n$, not necessarily equally likely. Then what property must the values $P(x_i)(= P(\{x_i\}))$ satisfy?

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Remark

These conditions lead us to the formal definition of probability.

Formal definition of probability

Definition

Suppose S is a sample space containing possible outcomes $x_1, \ldots, x_n$, not necessarily equally likely. If $P : S \to \mathbb{R}$ is a real-valued function on the sample space S satisfying

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then the **probability** $P(A)$ *of any event* A *is*

$$\sum_{x_i \in A} P(x_i).$$

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Remark

Now we can deal with outcomes with unequal possibilities.

Example: biased die

Example

A biased die has $P(1) = \frac{1}{3}$, $P(2) = P(3) = \frac{1}{12}$, *and* $P(4) = P(5) = P(6) = \frac{1}{6}$ *(here we write* $P(x)$ *for* $P(\{x\})$*). If the die is rolled once, find the probability that*

1. *an odd number appears;*
2. *the number that appears is less than* 3.

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1. *an odd number appears;*
2. *the number that appears is less than* 3*.*

Solution

(1) The event is $\{1, 3, 5\}$*, so the probability is* $P(1) + P(3) + P(5) = \frac{1}{3} + \frac{1}{12} + \frac{1}{6} = \frac{7}{12}$.

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1. *an odd number appears;*
2. *the number that appears is less than* 3.

Solution

(1) The event is $\{1, 3, 5\}$, *so the probability is*
$P(1) + P(3) + P(5) = \frac{1}{3} + \frac{1}{12} + \frac{1}{6} = \frac{7}{12}$.
(2) The event is $\{1, 2\}$, *so the probability is*
$P(1) + P(2) = \frac{1}{3} + \frac{1}{12} = \frac{5}{12}$.

The conditional probability

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Definition (Conditional probability)

Let A and B be events with $P(A) > 0$. The **conditional probability** *of B given A, denoted $P(B \mid A)$, is $\frac{P(B \cap A)}{P(A)}$.*

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Remark

Interpretation: given A, then we have a new sample space A, and a new event $B \cap A$.

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A fair coin is tossed 5 *times. Find the probability of obtaining at most* 1 *head, given that at least* 1 *head appears.*

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Solution

We need to find $P(B \mid A)$, where A is the event "at least 1 head appears" and B is the event "obtaining at most 1 head", then $B \cap A$ is "obtaining exactly 1 head".

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Solution

We need to find $P(B \mid A)$, *where* A *is the event "at least* 1 *head appears" and* B *is the event "obtaining at most* 1 *head", then* $B \cap A$ *is "obtaining exactly* 1 *head".* $P(B \cap A) = \binom{5}{1}/2^5 = \frac{5}{32}$.
And A^c *is the event "no head appears", so*
$P(A) = 1 - P(A^c) = 1 - \frac{\binom{5}{0}}{2^5} = \frac{31}{32}$.

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We need to find $P(B \mid A)$, where A is the event "at least 1 head appears" and B is the event "obtaining at most 1 head", then $B \cap A$ is "obtaining exactly 1 head". $P(B \cap A) = \binom{5}{1}/2^5 = \frac{5}{32}$. And A^c is the event "no head appears", so $P(A) = 1 - P(A^c) = 1 - \frac{\binom{5}{0}}{2^5} = \frac{31}{32}$. Then $P(B \mid A) = \frac{5}{31}$.

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Corollary

If S *consists of equally likely possible outcomes and* $P(B \mid A)$ *is well-defined, then* $P(B \mid A) = \frac{|B \cap A|}{|A|}$*.*

Independent events

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Recall the multiplication rule, the steps are independent to each other. We have an analogue in probability theory.

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Definition

Events A and B are **independent** *if $P(A \cap B) = P(A) \cdot P(B)$.*

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Definition

Events A and B are **independent** *if $P(A \cap B) = P(A) \cdot P(B)$.*

Proposition

Suppose A and B are events with $P(A) > 0$. They are independent if and only if $P(B \mid A) = P(B)$. In other words, the extra information of A does not affect the likelihood of B.

Example: defective light bulb

Example

Buymore Supermarket orders light bulbs from two suppliers, AA Electronics and AAA Electronics. It buys 30% *of its light bulbs from AA and* 70% *from AAA.* 2% *of the light bulbs bought from AA are defective, while* 3% *of the light bulbs bought from AAA are defective. Find the probability that a randomly selected light bulb*

1. *was purchased from AA and is defective;*
2. *is defective.*

Example: defective light bulb

Solution

Let A be the event "the light bulbs was bought from AA", B be the event "the light bulbs was bought from AAA", and C be the event "the light bulbs is defective". We are given that $P(A) = 0.3, P(B) = 0.7$, and $P(C \mid A) = 0.02, P(C \mid B) = 0.03$.

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Let A *be the event "the light bulbs was bought from AA",* B *be the event "the light bulbs was bought from AAA", and* C *be the event "the light bulbs is defective". We are given that* $P(A) = 0.3, P(B) = 0.7$*, and* $P(C \mid A) = 0.02, P(C \mid B) = 0.03$*.*
(1) This is $P(A \cap C)$*, which is*
$P(A) \cdot P(C|A) = 0.3 \cdot 0.02 = 0.006 = 0.6\%$*.*

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(1) This is $P(A \cap C)$, which is $P(A) \cdot P(C|A) = 0.3 \cdot 0.02 = 0.006 = 0.6\%$.

(2) This is $P(C)$. Since A and B are complementary to each other, we have $C = (C \cap A) \cup (C \cap B)$,

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(1) This is $P(A \cap C)$, which is $P(A) \cdot P(C|A) = 0.3 \cdot 0.02 = 0.006 = 0.6\%$.

(2) This is $P(C)$. Since A and B are complementary to each other, we have $C = (C \cap A) \cup (C \cap B)$, So

$$\begin{aligned} P(C) &= P(C \cap A) + P(C \cap B) \\ &= P(A)P(C \mid A) + P(B)P(C \mid B) \\ &= 0.3 \cdot 0.02 + 0.7 \cdot 0.03 \\ &= 0.027 = 2.7\%. \end{aligned}$$

A useful formula

Proposition

If $A_1, A_2, \cdots, A_n$ are mutually exclusive events with positive probability and their union is the entire sample space, then for any event X we have

$$P(X) = \sum_{i=1}^{n} P(A_i) P(X \mid A_i).$$

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$$P(X) = \sum_{i=1}^{n} P(A_i)P(X \mid A_i).$$

Proof.

BY the property of $A_1, A_2, \cdots, A_n$, we know that $X \cap A_i$'s are pairwise disjoint and their union is X.
So $P(X) = \sum_{i=1}^{n} P(X \cap A_i)$, and for each i, we have
$P(X \cap A_i) = P(A_i)P(X \mid A_i)$. □

Bayes's formula

The following formula is fundamental for inference.

Theorem (Bayes's formula)

Suppose events $A_1, A_2, \cdots, A_n$ *are mutually exclusive and* $\bigcup_{i=1}^{n} A_i$ *is the sample space* S *and* $P(A_i) > 0$ *for all* i. *For any event* X *with* $P(X) > 0$, *we have*

$$P(A_j \mid X) = \frac{P(A_j)P(X \mid A_j)}{P(X)},$$

where

$$P(X) = \sum_{i=1}^{n} P(A_i)P(X \mid A_i).$$

Homework Assignment #9

Section 7.3 Exercise 4(a)(c), 6(e), 13(b)(c), 25(a).
Section 7.4 Exercise 1(a), 4, 10(a)(d), 14, 20.