# Math 2603 - Lecture 17
# Section 7.5 & 7.6 Repetitions and derangements

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October 22nd, 2019

# Combinations with repeated elements

# Motivation: unordered selection with repetitions

So far we introduced ordered and unordered selections of distinct elements from a set. But in practice, we may need to select an element more than once.

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- A fair die is rolled $5$ times, the number of unordered outcomes.
- There are $10$ different types of bagels available at Dunkin Donuts, the number of ways to buy $6$ bagels.
- There are $13$ denominations in a deck of playing cards and $3$ cards are drawn, the number of unordered outcomes about their denominations.

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We need a systematic method for counting all these quantities.

# General model: objects in the boxes

**Remark**

*A general model of all these examples is: the number of ways to put $r$* **identical** *objects into $n$* **distinct** *(labeled) boxes.*

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*Example revisited:*

- $5$ *rolls distributed into* $6$ *distinct outcomes -* $r = 5, n = 6$.
- $6$ *bagels bought from* $10$ *distinct types -* $r = 6, n = 10$.
- $3$ *drawn cards from* $13$ *distinct denominations -* $r = 3, n = 13$.

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## Remark

*To distinguish objects and boxes, the key is whether they are identical (unordered) or not.*

# The formula

### Theorem

*The number of ways to put $r$ identical objects into $n$ distinct boxes is*

$$\binom{n+r-1}{r} = \binom{n+r-1}{n-1}.$$

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$$\binom{n+r-1}{r} = \binom{n+r-1}{n-1}.$$

**Remark**

*Feel free to directly apply this formula. The textbook does not prove it, but I will explain two different proofs, which provide different perspectives to understand it.*

# Equivalent characterizations

## Proposition

*The following sets have the same cardinality:*

1. *The set of all ways to put $r$ identical objects into $n$ distinct boxes.*
2. $\{(x_1, x_2, \ldots, x_n) \in \mathbb{Z}^n \mid x_i \geq 0, \sum_{i=1}^{n} x_i = r\}$.
3. $\{(y_1, y_2, \ldots, y_n) \in \mathbb{Z}^n \mid 0 \leq y_1 \leq y_2 \leq \cdots \leq y_n = r\}$.
4. $\{(z_1, z_2, \ldots, z_n) \in \mathbb{N}^n \mid \sum_{i=1}^{n} z_i = r + n\}$.
5. $\{(w_1, w_2, \ldots, w_{n-1}) \in \mathbb{N}^{n-1} \mid 1 \leq w_1 < w_2 < \cdots < w_{n-1} \leq n + r - 1\}$.

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(1) and (2): let $x_i$ be the numbers of objects in the $i$-th box. (2) and (3): let $y_i = \sum_{j=1}^i x_j$. (2) and (4): let $z_i = x_i + 1$, then $z_i \in \mathbb{N}$. (3) and (5): will prove in a while. $\square$

# An illustration

Suppose $r = 5, n = 6$. The five rolls of the die are $2, 6, 3, 3, 5$. The elements in the five versions are:

1. Unordered selection $2, 3, 3, 5, 6$.
2. $(0, 1, 2, 0, 1, 1)$ - $i$th entry is the number of appearance of $i$ among the five rolls;
3. $(0, 0+1, 0+1+2, 0+1+2+0, 0+1+2+0+1, 0+1+2+0+1+1) = (0, 1, 3, 3, 4, 5)$;
4. $(1, 2, 3, 1, 2, 2)$;
5. $(0+1, 1+2, 3+3, 3+4, 4+5) = (1, 3, 6, 7, 9)$.

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Each $n$ tuple $(z_1, z_2, \ldots, z_n)$ in the set corresponds to a unique way to divide these stars into $n$ parts, where each part contains at least one star. We can put $n - 1$ bars where each bar is between two adjacent stars

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How many places can we insert bars? $n + r - 1$. In addition, we can insert at most one bar at each place, so the number of ways to insert bars is $\binom{n+r-1}{n-1}$, which equals $\binom{n+r-1}{r}$.

# A proof by bijection

We haven't proved the equivalence of (3) and (5)

$$\{(y_1, y_2, \ldots, y_n) \in \mathbb{Z}^n \mid 0 \le y_1 \le y_2 \le \cdots \le y_n = r\}$$

$$\{(w_1, w_2, \ldots, w_{n-1}) \in \mathbb{N}^{n-1} \mid 1 \le w_1 < w_2 < \cdots < w_{n-1} \le n+r-1\}.$$

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**Remark**

*The cardinality of (5) is apparently "$n + r - 1$ choose $n - 1$".*

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*How many different collections of* $10$ *coins can be made from pennies, nickels, dimes and quarters?*

## Solution

*Coins are unordered, so* $r$ *is* $10$*. Denominations are distinct, so* $n$ *is* $4$*. By the formula, the answer is* $\binom{10+4-1}{4-1} = \binom{13}{3} = 286$*.*

# Example with constraints

### Example

*There are* $5$ *different flavors of pizza slices available at our food court. Suppose I need to buy* $6$ *slices for my colleagues and I am supposed to buy at least one slice of pepperoni pizza, how many ways can I do it?*

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## Solution

*Since I need to buy at least one slice of pepperoni pizza, I simply buy one slice first. Then the remaining $5$ slices are arbitrary! So essentially this is the case when $r = n = 5$, and the answer is $\binom{9}{4} = 126$.*

# Permutations with repeated elements

# Example: balls in $3$ colors

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*There are $6+3+2=11$ positions in total. First we choose $6$ positions for the red balls, we get $\binom{11}{6}$.*

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*There are $6+3+2=11$ positions in total. First we choose $6$ positions for the red balls, we get $\binom{11}{6}$. Next we choose $3$ positions from the remaining $11-6=5$ positions for the blue balls, we get $\binom{5}{3}$.*

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*There are $6+3+2=11$ positions in total. First we choose $6$ positions for the red balls, we get $\binom{11}{6}$. Next we choose $3$ positions from the remaining $11-6=5$ positions for the blue balls, we get $\binom{5}{3}$. Finally there is only one choice of white balls. The answer is $\binom{11}{6}\binom{5}{3}$.*

# Observing the pattern

**Remark**

*In this example, all the numbers we have are $x_1 = 6, x_2 = 3, x_3 = 2$, and we get $11 = x_1 + x_2 + x_3$. What about the answer in terms of them?*

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### Remark

*The answer is*

$$\binom{11}{6}\binom{5}{3} = \frac{11!}{6!(11-6)!} \cdot \frac{(11-6)!}{3!(11-6-3)!} = \frac{(x_1+x_2+x_3)!}{x_1!x_2!x_3!}.$$

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### Remark

*This is not a coincidence and actually it is the punchline instead.*

# The formula

**Theorem**

*Suppose there are $x_i$ identical objects of the $i$-th type, $1 \le i \le n$. Then the number of ways to arrange all these objects in an ordered row is*

$$\frac{(\sum_{i=1}^{n} x_i)!}{\prod_{i=1}^{n} (x_i!)}.$$

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**Proof.**

Regard all objects as distinct, then there are $(\sum_{i=1}^{n} x_i)!$ permutations in total. While for each above arrangement, for the $x_i$ identical objects of the $i$-th type, there are $x_i!$ ways to rearrange them. By the multiplication rule, there are $\prod_{i=1}^{n}(x_i!)$ permutations of "distinct" objects that correspond to the same arrangement. □

# Multinomial coefficients

## Corollary

*Suppose integers* $x_1, x_2, \ldots, x_n \geq 0$ *and their sum is* $m$. *Then in the expansion of* $\left(\sum_{i=1}^{n} a_i\right)^m$, *the coefficient of the monomial* $\prod_{i=1}^{n} a_i^{x_i}$ *is exactly*

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## Remark

*As a result,* $\frac{(\sum_{i=1}^n x_i)!}{\prod_{i=1}^n x_i!}$ *are called* **multinomial coefficients**.

# More example

**Example**

*In how many ways can the letters of the word "rearrange" be rearranged?*

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## Solution

*We keep track of the number of appearances of each letter.* $r$ *three times; both* $a$ *and* $e$ *twice;* $n$ *and* $g$ *once. So we have*

$$3, 2, 2, 1, 1.$$

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*The answer is*

$$\frac{(3+2+2+1+1)!}{3!2!2!1!1!} = \frac{9!}{3!2!2!} = \frac{9!}{24} = 15,120.$$

# Derangements

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*The only derangements are $231$ and $312$. Hence $D_3 = 2$.*

Question: what is the value of $D_n$?

# Solution using Inclusion-Exclusion

## Solution

*For $1 \leq i \leq n$, let $A_i$ be the set of all permutations on $1, 2, \ldots, n$ such that $i$ is at the $i$-th position. Then*

$$D_n = n! - |A_1 \cup A_2 \cup \cdots \cup A_n|.$$

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$$D_n = n! - |A_1 \cup A_2 \cup \cdots \cup A_n|.$$

*Now by the Principle of Inclusion-Exclusion, we can evaluate:*

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = \sum_{i=1}^{n} \left[ (-1)^{i+1} \cdot \sum_{j_1 < j_2 < \ldots < j_i} |A_{j_1} \cap \cdots \cap A_{j_i}| \right].$$

*Note that $A_{j_1} \cap \cdots \cap A_{j_i}$ is the set of permutations with $i$ numbers at their own positions, and other numbers at arbitrary positions. So the cardinality is $(n-i)!$.*

# Solution using Inclusion-Exclusion

## Solution

*Then we have*

$$\begin{aligned}
D_n &= n! - \sum_{i=1}^{n} \left[ (-1)^{i+1} \cdot \sum_{j_1<j_2<\ldots<j_i} (n-i)! \right] \\
&= n! - \sum_{i=1}^{n} \left[ (-1)^{i+1} \cdot \binom{n}{i} \cdot (n-i)! \right] \\
&= n! - \sum_{i=1}^{n} (-1)^{i+1} \frac{n!}{i!} \\
&= n! \sum_{i=2}^{n} (-1)^{i} \frac{1}{i!}.
\end{aligned}$$

# Example of $D_n$

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$$D_4 = 24 \cdot \left(\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right) = 12 - 4 + 1 = 9.$$

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## Example

$D_5 = 120 \cdot (\frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120}) = 60 - 20 + 5 - 1 = 44.$

# Asymptotic behavior of $D_n$

## Proposition

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**Corollary**

*When $n$ is sufficiently large, there are about $\frac{1}{e} \approx 37\%$ permutations are derangements.*

## Homework Assignment #10 - today

Section 7.5 Exercise 5, 6(b), 9, 11(b).
Section 7.6 Exercise 4, 5(b), 8.