Math 2603 - Lecture 18
Section 9.1 & 9.2 Graph Theory

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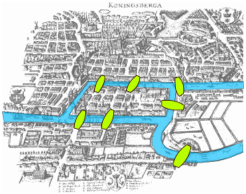
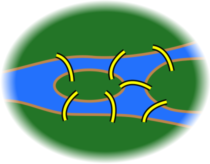
October 24th, 2019

Examples of graphs

Königsberg Bridge Problem





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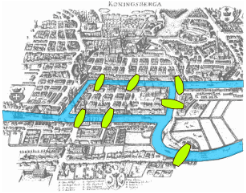

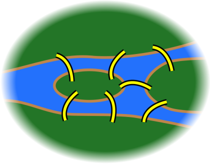

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In the 18-th century, Königsberg was the capital of East Prussia. (The city of Kaliningrad today, in an enclave of Russia, between Poland and Lithuania)

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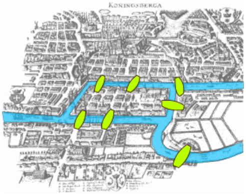

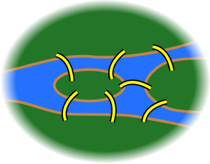

Example

In the 18*-th century, Königsberg was the capital of East Prussia. (The city of Kaliningrad today, in an enclave of Russia, between Poland and Lithuania) A river flowed through town and split into two branches, resulted in* 4 *land masses, connected by* 7 *bridges in total.*

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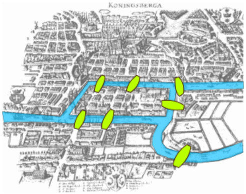

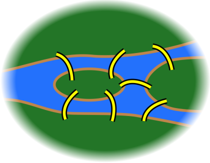

Example

In the 18-th century, Königsberg was the capital of East Prussia. (The city of Kaliningrad today, in an enclave of Russia, between Poland and Lithuania) A river flowed through town and split into two branches, resulted in 4 land masses, connected by 7 bridges in total. People wonder whether it's possible to start on one of the land masses, walk over each of the 7 bridges exactly once, and return to the starting point without getting wet.

The bridges at Georgia Tech

Maybe you have discovered, that these bridges are also on our campus!

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Remark

The answer is no.

The Three Houses-Three Utilities Problem

Example

There are three houses, each of which is to be connected to each of three utilities (water, electricity, and gas) by means of underground pipes. Is it possible to make these connections without any crossovers?

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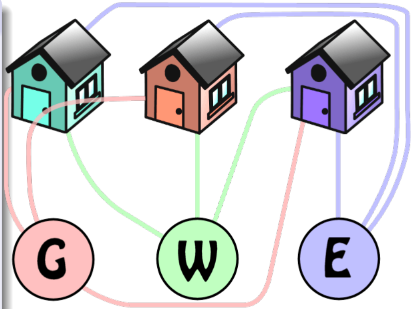

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How many pipes are needed? $3 \cdot 3 = 9$ *pipes.*
No matter how you make attempts, you can only draw 8 *pipes without any crossovers. And it turns out that the answer is no.*

Traveling Salesman Problem

Example

Given a list of cities and the distances between each pair of cities, what is the shortest possible route that visits each city and returns to the origin city?

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Remark

A special case is: is there a route that visits each city exactly once?

A summary of these examples

Each of them represents an important type of problems in graph theory, and we will elaborate later.

- Königsberg Bridge Problem - Eulerian circuits
- The Three Houses-Three Utilities Problem - Planar graphs
- Traveling Salesman Problem - Hamiltonian cycles

Remark

In summary, graphs are about the relationships among objects. (people, cities, computers, etc.)

Graph Theory

Definition of graphs

Definition

A **graph** *is a pair* $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ *of finite sets, where the set of* **vertices** $\mathcal{V}$ *is nonempty, and* $\mathcal{E}$ *is the set of* **edges**, *whose elements are sets of two distinct elements of* $\mathcal{V}$*. For example, if* $e \in \mathcal{E}$ *is an edge, then* $e = \{v, w\}$ *where* $v, w \in \mathcal{V}$ *are distinct vertices.*

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Definition

If $e = \{v, w\} \in \mathcal{E}$, *we say that* v *and* w *are* **adjacent** *and* e **connects/joins** *them.* v *and* w *are called the* **endpoints** *of* e. *Two edges are* **adjacent** *if they share a common endpoint.*

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Remark

These graphs are **undirected**. *If the edges are ordered pairs instead of sets, the graph would be* **directed**.

Example of a graph

Example

$\mathcal{V} = \{A, B, C, D\}, \mathcal{E} = \{\{A, B\}, \{B, C\}, \{C, D\}, \{A, C\}\}.$

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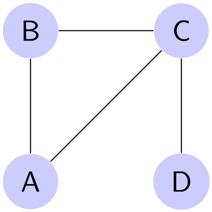

Loops and multiple edges

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A **loop** *is an edge whose two endpoints are the same vertex.*
Multiple edges *are more than one edges connecting the same two vertices.*

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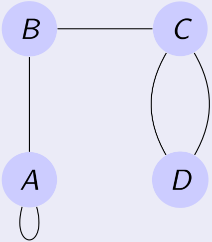
Example

In this example, there is a loop at vertex A and two multiple edges connecting C and D.



Simple graphs and pseudo graphs

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The graphs we just defined are **simple graphs** *- each edge connects two distinct vertices without a direction, there is at most one edge connect any two vertices.*

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If $\mathcal{E}$ is allowed to contain loop or multiple edges, the graph $\mathcal{G}$ is called a **pseudograph**. *In particular, simple graphs are pseudographs.*

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Example

The example on previous slide is not a simple graph but is a pseudograph.

Subgraphs

Definition

Graph $\mathcal{G}_1 = (\mathcal{V}_1, \mathcal{E}_1)$ *is a* **subgraph** *of graph* $\mathcal{G}_2 = (\mathcal{V}_2, \mathcal{E}_2)$*, if* $\mathcal{V}_1 \subseteq \mathcal{V}_2$ *and* $\mathcal{E}_1 \subseteq \mathcal{E}_2$*.*

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Example

$\mathcal{G}_1$ *is a subgraph of* $\mathcal{G}_2$.

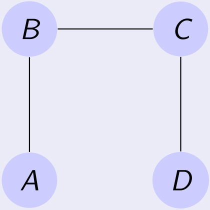

Figure: $\mathcal{G}_1$

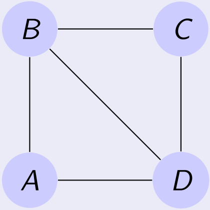

Figure: $\mathcal{G}_2$

Complete graphs

Definition

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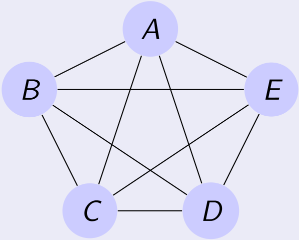
Example

The following graph is $\mathcal{K}_5$*:*



Bipartite graphs

Definition

A **bipartite graph** *is a graph* $(\mathcal{V}, \mathcal{E})$ *where* $\mathcal{V}$ *admits a partition into* $\mathcal{V}_1$ *and* $\mathcal{V}_2$ *such that every edge in* $\mathcal{E}$ *connects one vertex in* $\mathcal{V}_1$ *and another vertex in* $\mathcal{V}_2$*.*

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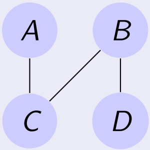
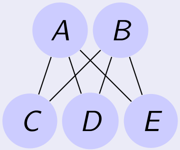
Example

Left: a bipartite graph but not complete. Right: $\mathcal{K}_{2,3}$





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Prove that any bipartite graph does not contain the subgraph $\mathcal{K}_3$ (triangle).

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Proof.

For any three vertices in the graph, since there are only two parts $\mathcal{V}_1, \mathcal{V}_2$, by the Pigeonhole Principle, there exist at least two vertices in the same part.

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For any three vertices in the graph, since there are only two parts $\mathcal{V}_1, \mathcal{V}_2$, by the Pigeonhole Principle, there exist at least two vertices in the same part.By definition of bipartite graphs, they are not connected by any edge, so there is no subgraph $\mathcal{K}_3$ with respect to these three vertices. $\square$

Degrees of vertices

Definition

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Solution

$\deg(A) = \deg(B) = 2, \deg(C) = 3, \deg(D) = 1$. *The sum is* $3 + 2 + 2 + 1 = 8 = 2 \cdot |\mathcal{E}|$.

Relationship between degrees and # of edges

There is a relationship between the degrees and the number of edges in a graph.

Proposition (Euler)

The sum of the degrees of all vertices of a pseudograph is twice the number of edges and hence an even integer.

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Proof.

Each edge contributes 2 is both the left hand side and the right hand side of the formula. $\square$

Degree sequences

Definition

Suppose $d_1 \geq d_2 \geq \cdots \geq d_n$ *are all degrees of vertices of a pseudograph* $\mathcal{G}$*, then* $d_1, d_2, \cdots, d_n$ *is called the* **degree sequence** *of* $\mathcal{G}$.

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Solution

$3, 2, 2, 1.$

Homework Assignment #10 - today

Section 9.1 Exercise 3.
Section 9.2 Exercise 3, 5, 6, 13, 19, 21(b).