Math 2603 - Lecture 19
Section 9.3 Isomorphism of Graphs

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Motivation

Different visualization of the same graph

Remark

As we explained last time, the essential feature of a graph is the incidental relationship among its vertices. Therefore, there are different visualization of the same graph.

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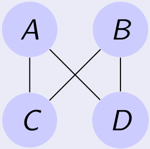
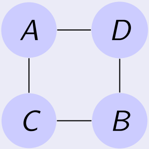
Example

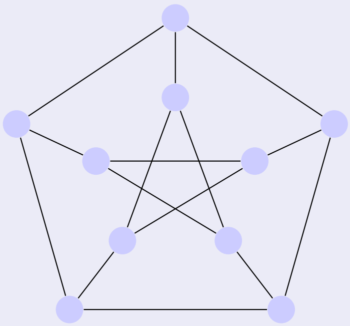
These two figures illustrate the same graph.





Example: Petersen graph

Example



What about these two graphs?

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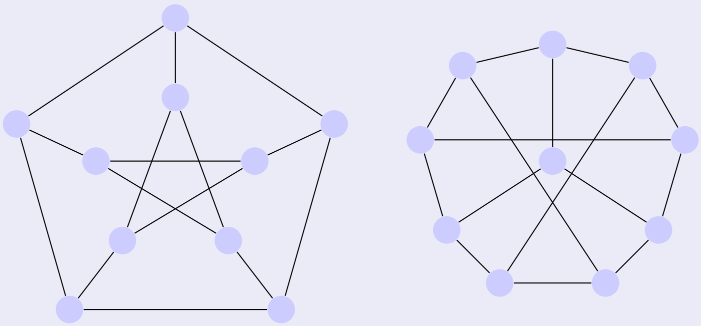

What about these two graphs? They are both the Petersen graph.

How to describe “equal”

Remark

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Graph Isomorphism

Definition

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Given graphs $\mathcal{G}_1 = (\mathcal{V}_1, \mathcal{E}_1)$ and $\mathcal{G}_2 = (\mathcal{V}_2, \mathcal{E}_2)$, we say that $\mathcal{G}_1$ is **isomorphic** *to $\mathcal{G}_2$ and write $\mathcal{G}_1 \cong \mathcal{G}_2$, if and only if there is a one-to-one function $\varphi : \mathcal{V}_1 \to \mathcal{V}_2$ that induces a one-to-one function $\mathcal{E}_1 \to \mathcal{E}_2$ as well. In other words,*

- *if vw is an edge in $\mathcal{E}_1$, then $\varphi(v)\varphi(w)$ is also an edge in $\mathcal{E}_2$;*
- *every edge in $\mathcal{E}_2$ is of the form $\varphi(v)\varphi(w)$ where $vw \in \mathcal{E}_1$.*

We call φ an **isomorphism** *from $\mathcal{G}_1$ to $\mathcal{G}_2$. Abusing of notation, we also say $\varphi : \mathcal{G}_1 \to \mathcal{G}_2$ is an isomorphism.*

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Sketch of proof.

Reflexivity: the identity map works.
Symmetry: if $\varphi : \mathcal{G}_1 \to \mathcal{G}_2$ is an isomorphism, so is $\varphi^{-1} : \mathcal{G}_2 \to \mathcal{G}_1$.
Transitivity: if $\varphi_1 : \mathcal{G}_1 \to \mathcal{G}_2$ and $\varphi_2 : \mathcal{G}_2 \to \mathcal{G}_3$ are both isomorphisms, then $\varphi_2 \circ \varphi_1$ is an isomorphism from $\mathcal{G}_1$ to $\mathcal{G}_3$. □

Necessary conditions for graph isomorphism

Proposition

If $\mathcal{G}_1$ and $\mathcal{G}_2$ are isomorphic graphs, then they have the

- *same number of vertices;*
- *same number of edge;*
- *same degree sequences.*

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Remark

This proposition is very useful to identify non-isomorphic graphs.

Example: which graphs are isomorphic?

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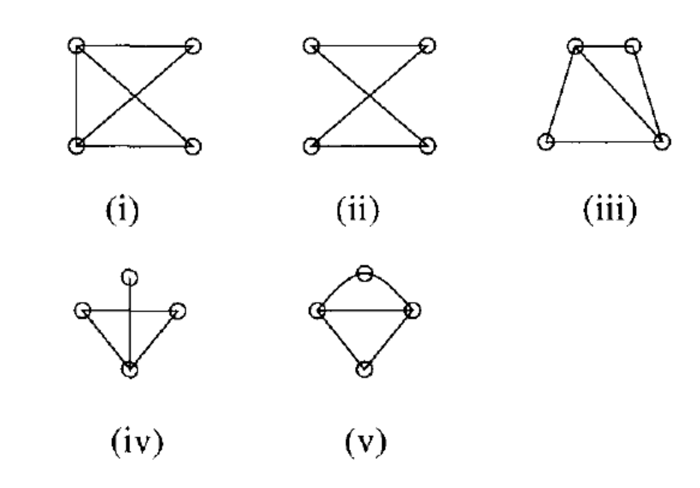
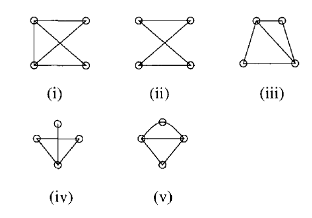


Example: which graphs are isomorphic

Solution

Graph	V	E	Deg
(i)	4	5	3, 3, 2, 2
(ii)	4	4	2, 2, 2, 2
(iii)	4	5	3, 3, 2, 2
(iv)	4	4	3, 2, 2, 1
(v)	4	5	3, 3, 2, 2



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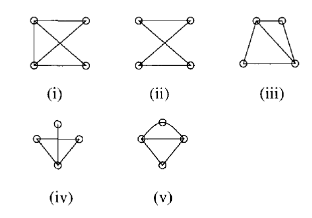

It turns out that $(i), (iii), (v)$ *are all isomorphic, and* (ii) *and* (iv) *are not isomorphic to any other one.*

More properties preserved by graph isomorphisms

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There are a lot more properties of graphs that are preserved by graph isomorphisms. As a result, if two graphs have different status with respect to some property, then they are not isomorphic.

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- *connectivity;*
- *length of the longest path/cycle;*
- *whether it has an Eulerian path/circuit;*
- *whether it has a Hamiltonian path/cycle;*
- *whether it is planar.*

We will elaborate them in the next few weeks.

Homework Assignment #11

Section 9.3 Exercise 3(a), 5(b), 8, 11(a).