Math 2603 - Lecture 2
Chapter 1 - Logic

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Truth tables

Motivation

For statements, we care about whether they are true or false. If we know the truthfulness of the atom statements in a compound statement, we should be able to know the truthfulness of the compound statement.

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When the compound statement is complicated, we need a systematic way to do that.

Truth values and truth tables

Definition

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Given a statement p, if the truth values of the components of p are determined, then the truth value of p is also determined. This relationship is characterized in **truth tables**.

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Remark

These non-compound statements appeared a compound statement are called **statement variables**.

Truth table of negation

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p	$\neg p$
T	F
F	T

Figure: The truth table of negation $\neg$.

Truth table of conjunction

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p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Figure: The truth table of conjunction $\wedge$.

Truth table of disjunction

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p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Figure: The truth table of disjunction $\vee$.

Truth table of implication

We also have the truth table of $p \to q$.

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p	q	$p \to q$
T	T	T
T	F	F
F	T	$\mathbf{T}$
F	F	$\mathbf{T}$

Figure: The truth table of implication $\to$.

Truth tables for more general statements

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$$(p \vee \neg q) \wedge (\neg p \vee r),$$

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we need to write a truth table with more rows. And we can add columns for the intermediate statements.
In this case, we can write the following table

p	q	r	$\neg p$	$\neg q$	$p \vee \neg q$	$\neg p \vee r$	$(p \vee \neg q) \wedge (\neg p \vee r)$
T	F	T	F	T	T	T	T
				$\dots$			

Figure: The truth table of $(p \vee \neg q) \wedge (\neg p \vee r)$.

Tautologies and contradictions

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A **contradiction** *is a statement that is always false regardless of the truth values of its statement variables.*

Examples of tautologies and contradictions

Example

Show that $p \vee \neg p$ *is a tautology and* $p \wedge \neg p$ *is a contradiction.*

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Show that $p \vee \neg p$ *is a tautology and* $p \wedge \neg p$ *is a contradiction.*

Proof.

Simply by truth tables.

p	$\neg p$	$p \vee \neg p$
T	F	T
F	T	T

p	$\neg p$	$p \wedge \neg p$
T	F	F
F	T	F

$\square$

Properties in propositional logic

Logical equivalence

Definition

Two statements p and q are **logically equivalent**, *denoted by $p \Leftrightarrow q$, if and only if the following conditions are satisfied:*

1. *for every combination of the truth value of these statement variables, p and q have the same truth value.*

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Remark

One method to check whether p and q are logically equivalent is to write their truth tables.

Checking logical equivalences by truth tables

Example

Show that statements $p \wedge \neg q$ *and* $\neg(\neg p \vee q)$ *are logically equivalent.*

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Example

Show that statements $p \wedge \neg q$ and $\neg(\neg p \vee q)$ are logically equivalent.

Proof.

Their truth tables are identical:

p	q	$\neg q$	$p \wedge \neg q$
T	T	F	F
T	F	T	T
F	T	F	F
F	F	T	F

p	q	$\neg p$	$\neg p \vee q$	$\neg(\neg p \vee q)$
T	T	F	T	F
T	F	F	F	T
F	T	T	T	F
F	F	T	T	F

$\square$

Examples about logical equivalences

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1. Idempotence: $(p \wedge p) \Leftrightarrow (p \vee p) \Leftrightarrow p$.
2. Double Negation: $\neg(\neg p) \Leftrightarrow p$.
3. Commutativity: $(p \wedge q) \Leftrightarrow (q \wedge p)$; $(p \vee q) \Leftrightarrow (q \vee p)$.
4. Associativity: $(p \wedge (q \wedge r)) \Leftrightarrow ((p \wedge q) \wedge r)$; $(p \vee (q \vee r)) \Leftrightarrow ((p \vee q) \vee r)$.
5. Distributivity: $(p \vee (q \wedge r)) \Leftrightarrow ((p \vee q) \wedge (p \vee r))$; $(p \wedge (q \vee r)) \Leftrightarrow ((p \wedge q) \vee (p \wedge r))$.

De Morgan's Laws

Augustus De Morgan (1806-1871) was a British mathematician and logician.

Definition

De Morgan's Laws *consist of the following two pairs of logically equivalent statements:*

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Remark

We can easily verify De Morgan's Laws using truth tables.

Example: converse and inverse are logically equivalent

Recall, the converse of $p \to q$ is $q \to p$.

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Definition

The **inverse** *of an implication* $p \to q$ *is* $\neg p \to \neg q$.

Example

Show that $q \to p$ *and* $\neg p \to \neg q$ *are logically equivalent.*

Converse and inverse are contrapositive

Proof.

Their truth tables are identical:

p	q	$q \to p$
T	T	T
T	F	T
F	T	F
F	F	T

p	q	$\neg p$	$\neg q$	$\neg p \to \neg q$
T	T	F	F	T
T	F	F	T	T
F	T	T	F	F
F	F	T	T	T

$\square$

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p	q	$\neg p$	$\neg q$	$\neg p \to \neg q$
T	T	F	F	T
T	F	F	T	T
F	T	T	F	F
F	F	T	T	T

$\square$

Remark

The converse and inverse of an implication are also contrapositive.

A cautionary example

Example

Consider the following compound statement α: 'Jane is tall and thin'. What is its negation in English sentences?

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Solution

Let p be 'Jane is tall' and q be 'Jane is thin'. Then a is simply $p \wedge q$. By De Morgan's Laws, $\neg a$ is logically equivalent to $(\neg p) \vee (\neg q)$, which is 'Jane is not tall or not thin'. However, if we take the negation of the original sentence, it becomes 'Jane is not tall and thin'. How to understand this phenomenon?

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The negation of a is 'Jane is not (tall and thin)'. Actually this $\neg a$ is not 'evaluated' at all. And if we do want to break it down, we apply De Morgan's Laws and get the correct result.

Logical arguments

Motivation

Last time I talked about proof techniques. These techniques represent general patterns of proof, for example, proof by cases works as follows:

$$p \to r$$

$$q \to r$$

$$p \vee q$$

Therefore r.

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$$p \to r$$
$$q \to r$$
$$p \vee q$$
Therefore r.

You may agree with me that, this kind of patterns "makes sense". In this section, we discussion a variety of these patterns so that we have more tools to write proofs.

Logical arguments

Definition

An **argument** *is a sequence of statements. All statements in an argument, except for the final one, are called* **premises** *or* **hypotheses**. *The final statement is called the* **conclusion**.

Valid and invalid arguments

Definition

An argument is called **valid** *if, when all premises are true, then the conclusion is also true. Otherwise it is called* **invalid***.*

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Remark

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Remark

Equivalently, an argument is valid if and only if the statement 'the conjunction of its premises implies its conclusion' is a tautology.

Example: check validity of arguments using truth table

Using truth table, it suffices to check the rows where all premises are true. These rows are called **critical rows**.

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Example

Decide whether the following argument is valid:

If I like math, then I will study it.
I study math or I fail the course.

If I fail the course, then I don't like math.

Example: check validity of arguments using truth table

Using truth table, it suffices to check the rows where all premises are true. These rows are called **critical rows**.

Example

Decide whether the following argument is valid:

$$\begin{array}{c} \textit{If I like math, then I will study it.} \\ \textit{I study math or I fail the course.} \\ \hline \textit{If I fail the course, then I don't like math.} \end{array}$$

Hint

Let p be "I like math", q be "I study math", and r be "I will fail the course". The argument becomes

$$\begin{array}{c} p \to q \\ q \vee r \\ \hline r \to \neg p. \end{array}$$

Solution of example

Solution

We write the truth table:

			Premise #1	Premise #2		Conclusion
p	q	r	$p \to q$	$q \vee r$	$\neg p$	$r \to \neg p$
T	T	T	$\mathbf{T}$	$\mathbf{T}$	F	$\mathbf{F}$
T	F	T	F	T	F	F
F	T	T	T	T	T	T
F	F	T	T	T	T	T
T	T	F	T	T	F	T
T	F	F	F	F	F	T
F	T	F	T	T	T	T
F	F	F	T	F	T	T

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T	T	T	$\mathbf{T}$	$\mathbf{T}$	F	$\mathbf{F}$
T	F	T	F	T	F	F
F	T	T	T	T	T	T
F	F	T	T	T	T	T
T	T	F	T	T	F	T
T	F	F	F	F	F	T
F	T	F	T	T	T	T
F	F	F	T	F	T	T

By the first row, the argument is not valid.

Rules of inference

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1. Modus ponens;
2. Modus tollens;
3. Disjunctive syllogism;
4. Chain rule;
5. Resolution.

Modus ponens

The most fundamental rule of inference is modus ponens.

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Definition

Modus ponens *is the following valid argument:*

$$\begin{array}{c} p \\ p \to q \\ \hline q \end{array}$$

Remark

In Latin, 'modus ponens' means 'method of affirming'. And it is indeed an important way to affirm a statement.

Validity of modus ponens

Even though it is so straight forward, we still need to verify the validity of modus ponens. We can write its truth table

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Proof.

	Premise #1	Premise #2	Conclusion
q	p	$p \to q$	q
T	T	T	T
T	F	T	T
F	T	F	F
F	F	T	F

The only critical row is the first row, and modus ponens is valid. $\square$

Modus tollens

Definition

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Remark

Since $p \to q$ *is logically equivalent to its contrapositive* $\neg q \to \neg p$*, its validity follows from modus ponens.*

Disjunctive syllogism

Definition

Disjunctive syllogism *is the following valid argument:*

$$\begin{array}{c} p \vee q \\ \neg p \\ \hline q \end{array}$$

Chain rule

Definition

Chain rule *is the following valid argument:*

$$\begin{array}{c} p \to q \\ q \to r \\ \hline p \to r \end{array}$$

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Chain rule *is the following valid argument:*

$$\begin{array}{c} p \to q \\ q \to r \\ \hline p \to r \end{array}$$

Remark

Chain rule is related to modus ponens.

Resolution

Definition

Resolution *is the following valid argument:*

$$\begin{array}{c} p \vee r \\ q \vee \neg r \\ \hline p \vee q \end{array}$$

Resolution

Definition

Resolution *is the following valid argument:*

$$\begin{array}{c} p \vee r \\ q \vee \neg r \\ \hline p \vee q \end{array}$$

Remark

Validity: we prove by contradiction. Suppose premises are true and $p \vee q$ *is false, then both* p *and* q *are false. Since* $p \vee r$ *is true,* r *must be true; since* $q \vee \neg r$ *is true,* $\neg r$ *must be true, a contradiction!*

HW Assignment #1 - Chapter 1

Section 1.1 Exercise 1(a)(c), 3, 9.
Section 1.2 Exercise 2(a), 3(b), 4.
Section 1.3 Exercise 3(a)(b), 5(b)(c)(f)(j).