Math 2603 - Lecture 20
Section 10.1 & 10.2 Eulerian circuits and Hamiltonian cycles

Bo Lin

November 5th, 2019

Eulerian circuits

Motivation: walking on a graph

Remark

One motivation of graphs is the abstraction of transportations - vertices are places (cities, countries, locations, etc.), and edges are roads, routes. We care about how to get one place to another, and also the number of ways, the shortest distance.

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So we want to define such concepts on graphs and pseudographs.

Walks, trails, circuits, and cycles

Definition

In pseudographs, a **walk** *is an alternating sequence of vertices and edges ($n \in \mathbb{N}$)*

$$v_1, e_1, v_2, e_2, \cdots, v_n, e_n, v_{n+1},$$

such that for $i = 1, 2, \ldots, n$, the edge e_i has two endpoints v_i and v_{i+1}. The **length** *of such a walk is n, the number of edges in it. A walk is* **closed** *if $v_1 = v_{n+1}$.*

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Definition

A **trail** *is a walk where all edges in it are distinct. A* **path** *is a walk where all vertices in it are distinct. A* **circuit** *is a closed trail. A circuit with only $v_1 = v_{n+1}$ appearing twice and all other v_i each appearing once is a* **cycle**. *An n-***cycle** *is a cycle with length n.*

Summary

Terms	Features
Walk	Alternating sequence of vertices and edges
Trail	Walk with distinct edges
Path	Walk with distinct vertices
Circuit	Closed trail
Cycle	Closed trail with distinct vertices

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Proposition

A path is also a trail.

Remark

When addressing walks, it is enough to list the vertices in order.

Eulerian circuits

Definition

An **Eulerian circuit** *in a pseudograph is a circuit that contains every vertex and every edge. A pseudograph is* **Eulerian** *if it contains a Eulerian circuit.*

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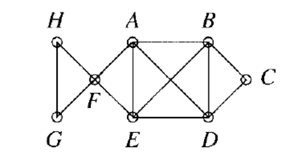
Remark

Königsburg Bridge Problem could be paraphrased as: is that graph with 4 *vertices and* 7 *edges Eulerian?*

Example: Eulerian

Example

1. *Is the walk $ABCDEFGHFA$ a circuit?*
2. *Is it an Eulerian circuit?*
3. *Is this graph Eulerian?*



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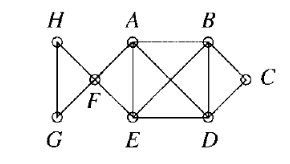

Solution

(1) Yes.

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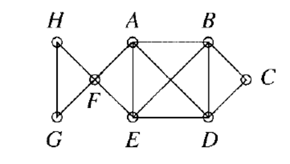

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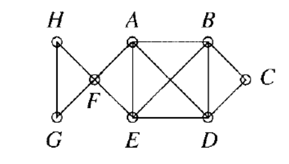

Solution

(1) Yes. (2) No, because edges including BD are not included in the circuit. (3) Yes, $ABCDEFGHFADBEA$ is an Eulerian circuit.

Connectivity

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Connected-ness is an important feature of pseudographs. If a pseudograph is not connected, it consists of **connected components**.

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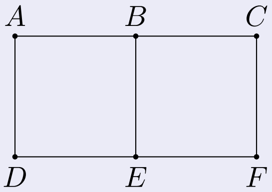
Proposition

Every Eulerian pseudograph is connected.

Condition on degree sequences

Example

1. *What is the degree sequence of this graph?*
2. *Is this graph Eulerian?*



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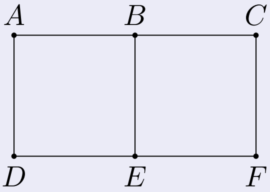

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(1) $3, 3, 2, 2, 2, 2.$

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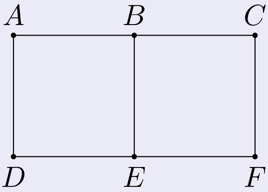

Solution

(1) $3, 3, 2, 2, 2, 2.$ *(2) No.*

Necessary condition is also sufficient

Remark

Fix a vertex A*.* A *may appear several times in an Eulerian circuit. But each appearance would cover* 2 *edges connecting* A*. Since every edge appears exactly once in an Eulerian circuit, we draw the conclusion that* $2 \mid \deg(A)$*! So the degrees of all vertices in an Eulerian pseudograph must be even.*

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It's perfect that the necessary conditions are also sufficient.

Theorem

A pseudograph is Eulerian if and only if it is connected and the degree of every vertex is an even number.

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Theorem

A pseudograph is Eulerian if and only if it is connected and the degree of every vertex is an even number.

Remark

The degree sequence of the Königsburg Bridges graph is $5, 3, 3, 3$, so no solution.

Idea of the proof

Sketch of proof.

It suffices to prove that any connected pseudographs with all degrees of vertices even must be Eulerian. We start from any vertex A. If A only has loops, the pseudographs only has one vertex and we are done.

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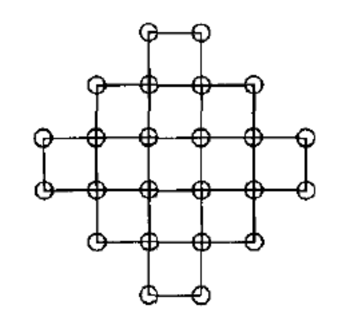
An illustration

Example

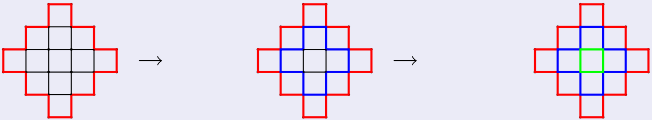
Find an Eulerian circuit of the following graph:



An illustration

Solution

We find the circuits one by one:



Then we concatenate the red, blue, and green circuits to make an Eulerian circuit.

Characterization of Eulerian trails

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Key observation: if one adds an extra edge connecting the first and last vertices in an Eulerian trail, then the trail becomes an Eulerian circuit!

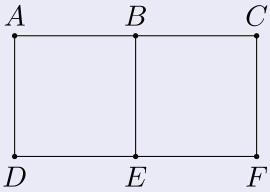
Theorem

A pseudograph $\mathcal{G}$ possesses an Eulerian trail between two distinct vertices u, v if and only if $\mathcal{G}$ is connected and the degrees of all vertices other than u, v are even.

Example: find an Eulerian trail

Example

1. *Find an Eulerian trail of this graph between B and E.*



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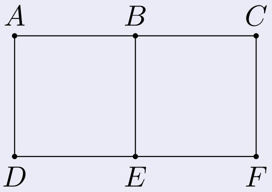

Solution

$$BADEFCBE.$$

Hamiltonian cycles

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Eulerian circuits require that all edges are distinct. What about vertices being distinct?

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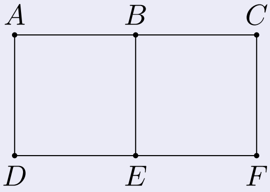
Remark

Hamiltonian graphs are named after Sir William Rowan Hamilton (1805-1865) from Ireland. They have applications in operational research problems. For example, postmen prefer to visit each address exactly once in a network.

Example revisited

Example

1. *Is this graph Hamiltonian?*



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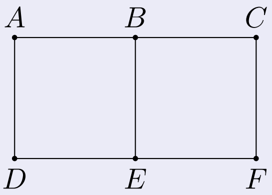

Solution

Yes. $ABCFEDA$ *is a Hamiltonian cycle.*

A basic property

Proposition

1. *If* $\mathcal{H}$ *is a Hamiltonian cycle in graph* $\mathcal{G}$*, and a vertex* v *has degree* 2*, then both edges incident with* v *must be part of* $\mathcal{H}$*.*
2. $\mathcal{H}$ *does not contain any shorter cycle.*

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Proposition

1. *If $\mathcal{H}$ is a Hamiltonian cycle in graph $\mathcal{G}$, and a vertex v has degree 2, then both edges incident with v must be part of $\mathcal{H}$.*
2. *$\mathcal{H}$ does not contain any shorter cycle.*

Remark

(1) is useful, either to justify the non-existence of Hamiltonian cycles, or fix part of the cycle and narrow down our search space.

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If a graph has $n \geq 3$ vertices and every vertex has degree at least $\frac{n}{2}$, then it is Hamiltonian.

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Sketch of proof.

We pick a longest path $v_1, v_2, \ldots, v_t$. By the condition on degrees, we can find i such that $v_1 v_{i+1}$ and $v_t v_i$ are both edges.

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Example: application of Dirac's Theorem

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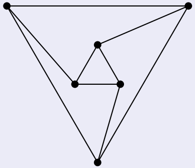

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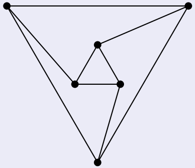

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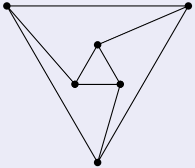

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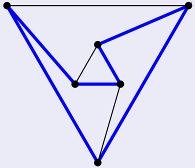

Homework Assignment #12 - today

Section 10.1 Exercise 4(b)(d)(e), 5, 6, 12(a)(b), 17.
Section 10.2 Exercise 3(d), 4, 11(a)(b), 24.