Math 2603 - Lecture 22
Section 12.1 & 12.2 Trees

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November 12th, 2019

Trees

Motivation

There are many examples of graphs in application that look like an ordinary tree. For instance,

- The hierarchy of an organization (company, schools, troops).
- The folders in a computer system.
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Remark

We have an abstraction of all these graphs, which is called **trees***.*

Definition

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A **tree** *is a connected graph without circuits.*

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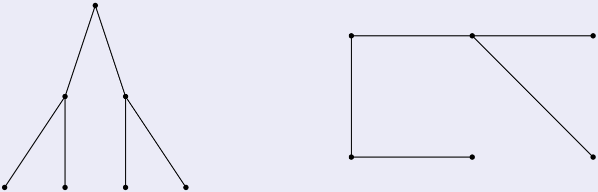
Example

The following graphs are trees.



Equivalent characterization

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Proposition

Let $\mathcal{G}$ be a graph. The following statements are equivalent.

1. *$\mathcal{G}$ is a tree;*
2. *$\mathcal{G}$ is connected and* **acyclic** *(without cycles);*
3. *between any two vertices of $\mathcal{G}$, there is a unique path connecting them.*

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Sketch of proof.

(1) $\to$ (2): cycles are circuits. (2) $\to$ (3): "connected" implies the existence, "acyclic" implies at most one path. (3) $\to$ (1): By 10.1 Ex. 15, circuits contain cycles and a cycle implies two paths. $\square$

Roots and leaves

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Remark

When we draw rooted trees, we usually put the root on top, and all edges have directions from top to bottom.

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A **leaf** *is a vertex in trees with degree* 1.

Existence of leaf

Lemma

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Since each vertex has degree at least 2, we can keep extend a path, until we reach a vertex that we have already visited, and then we end up with a circuit. $\square$

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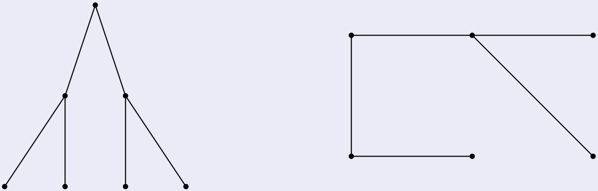
Corollary

Every tree has at least one leaf.

Number of edges and vertices

Remark

Recall the previous example



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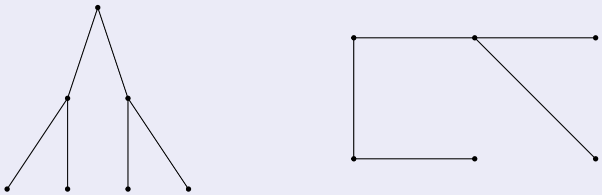

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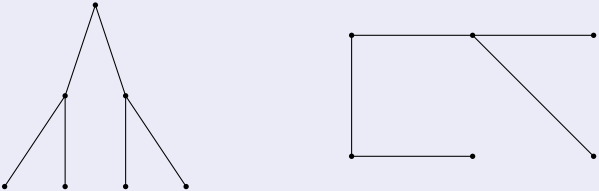

What are the number of vertices and edges of the trees? 7 *vertices,* 6 *edges;* 6 *vertices,* 5 *edges. This is not a coincidence.*

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"If": it suffices to show that the graph does not contain a circuit. If there is a circuit, then there is a cycle too. Delete one edge from the cycle, the graph is still connected. If the remaining graph still has a circuit, we do the same thing again, until there is no more circuit. Then we obtained a tree with n vertices, while we deleted at least one edge, so the tree has at most $n - 2$ edge, a contradiction to the "only if" part. $\square$

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Proof.

If there is a unique leaf instead, the total sum of degrees is at least $2 \cdot (n-1) + 1 = 2n - 1 > 2n - 2 = 2 \cdot |\mathcal{E}|$, a contradiction to Euler's formula. $\square$

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Any edge added to a tree must produce a cycle.

Proof.

The number of edges means that the new graph is no longer a tree, but it is still connected, so it admits a circuit, and thus a cycle. □

Spanning Trees

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Remark

This concept is motivated from applications - for example, to establish a postal network with least weight.

Number of spanning trees

Remark

We would like to count the number of all spanning trees in the given graph. This is solved by German physicist Gustav Kirchhoff (1824-1887), which involves the **adjacency matrix** *of a graph and the* **cofactors** *of matrices. As a result, we skip the Kirchhoff's Theorem.*

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Remark

Instead we consider a special case - the number of all spanning trees in $\mathcal{K}_n$. Note that this is equivalent to the number of labeled trees on n vertices.

Cayley's formula

Theorem

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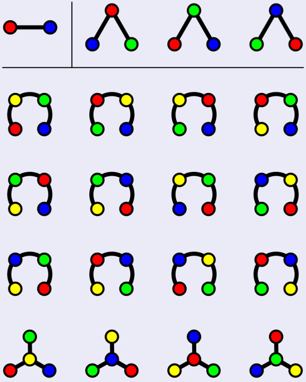
Example

All labeled trees with $n = 2, 3, 4$.



An elegant proof

We introduce an elegant proof only using bijection - we count the same quantity in two different ways and build the equality we want.

Proof.

We consider the number of ordered sequences of the $n-1$ directed edges of rooted labeled trees with n vertices. For example, if the root is red, and two edges $red - blue$ and $red - green$, then there are $P(2,2) = 2! = 2$ such ordered sequences of edges in this labeled tree.

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There is another way to count the number. We begin with an empty graph, and we choose the directed edges one by one. Suppose we already chose $n-k$ directed edges, the graph would be a union of k rooted trees. For the next directed edge, its starting vertex could be any of the n vertices, and its target can only be one of the roots in the other $k-1$ rooted trees. So in this step, we have $n(k-1)$ choices.

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$$\prod_{k=2}^{n} n(k-1) = n^{n-1} \cdot (n-1)! = n^{n-2} \cdot n!.$$

Hence $T_n \cdot n! = n^{n-2} \cdot n!$, $T_n = n^{n-2}$. $\square$

Homework Assignment #13 - today

Section 12.1 Exercise 1, 8, 16, 18, 24.

Section 12.2 Exercise 6, 8, 11.