Math 2603 - Lecture 24
Section 13.1 Planar graphs

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November 19th, 2019

Planar Graphs

Recall: motivation

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Remark

There are also practical applications, like in the Three Houses-Three Utilities Problem.

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Remark

Note that the definition is existential - as long as there exists one way to draw the graph without crossing edges, then it is planar. As a result, if one can't do it in a few attempt, one still cannot draw the conclusion that the graph is not planar.

Example: $\mathcal{K}_4$

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Is the complete graph with 4 vertices $\mathcal{K}_4$ planar?

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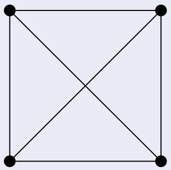
Example

Is the complete graph with 4 *vertices* $\mathcal{K}_4$ *planar?*

Solution



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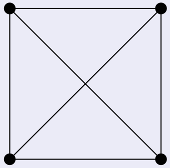

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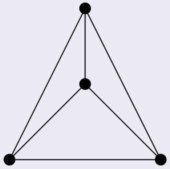

This plot implies that $\mathcal{K}_4$ is planar

Example: triangular prism

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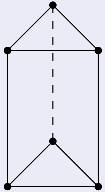
The following graph is of the triangular prism. Is it planar?



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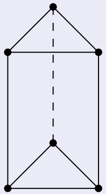

Solution

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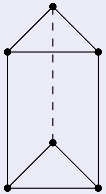

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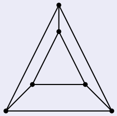
Yes. If we enlarge the base triangle a little bit, and project the top triangle onto the plane, we get the following graph which is planar.



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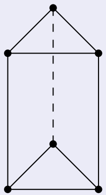

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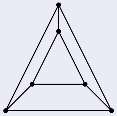

Remark

Then we have natural questions: are there graphs that are not planar? What is the characterization of planar graphs?

Euler's Theorem

Regions

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Example

Trees only have one region, which is the exterior region.

Connections to polyhedra

Remark

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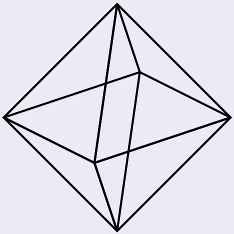
Proposition

For polyhedra, we can project down/expand it and convert it into a planar graph. The number of vertices and edges remain the same, and the number of regions is the number of faces.

Example: planar graphs of polyhedra

Remark

We have seen the projection of tetrahedron and triangular prism. Another example would be the bi-pyramid:



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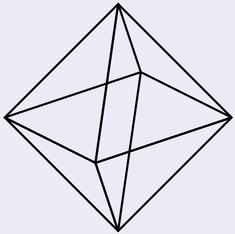

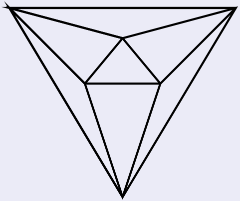

Examples: number of vertices, edges and regions

Euler observed a pattern among the three numbers:

Polyhedron	Vertices	Edges	Faces	$V + F$
Cube	8	12	6	14
Triangular Prism	6	9	5	11
Tetrahedron	4	6	4	8
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Remark

In all these examples, $V+F-E=2$, what an amazing pattern!

Another formula by Euler

Theorem (Euler's Theorem)

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Corollary

Let $\mathcal{G}$ be a connected plane graph with V vertices, E edges and R regions, then

$$V - E + R = 2.$$

Proof of Euler's Theorem

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A necessary condition for planar graphs

Theorem

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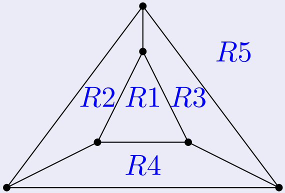
$$6 = 3V - 3E + 3R \leq 3V - 3E + 2E = 3V - E.$$

□

Illustration of the number N

Example



Triangular Prism

Region	*# edges*
$R1$	3
$R2$	4
$R3$	4
$R4$	4
$R5$	3
N	18

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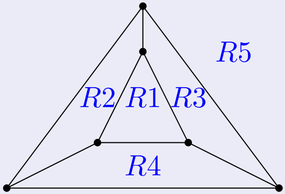

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$$E = 9, 18 = 2E = N > 15 = 3R.$$

Conclusions

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Proof.

$V = 5$ and $E = \binom{5}{2} = 10 > 9 = 3V - 6$. □

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Homeomorphic graphs

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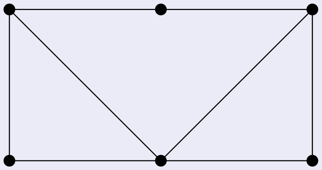
Definition

Two graphs are **homeomorphic** *if and only if both can be obtained from the same graph by adding (degree 2) vertices to edges.*

Examples

Example

Both graphs below are homeomorphic:



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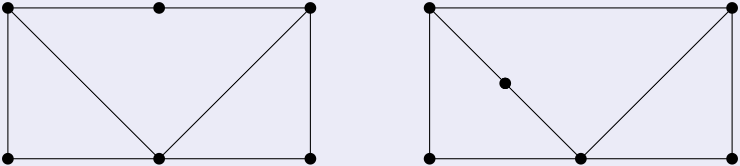

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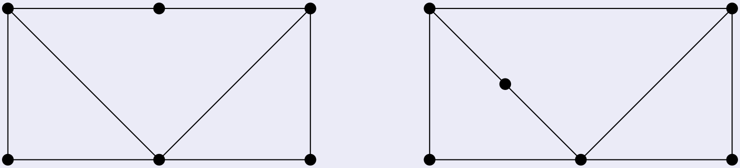

Remark

Homeomorphism preserves planar property.

Kuratowski's Theorem

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Theorem (Kuratowski's Theorem)

A graph is planar if and only if it has no subgraph homeomorphic to $\mathcal{K}_{3,3}$ or $\mathcal{K}_5$.

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Remark

Next time, we will see a related topic - coloring of graphs and the Four color Theorem.

Homework Assignment #14 - today

Section 13.1 Exercise 4, 8, 11, 19.