Math 2603 - Lecture 26
Section 14.1 & 14.2
The Max Flow-Min Cut Theorem

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November 26th, 2019

Flows and cuts

Motivation

In reality, we have various networks for transportation. For example:

- Gas & Gasoline;
- Trains & Flights;
- Electricity;
- Telephone calls.

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Remark

There are two notable features of these networks: many edges are directed, and each edge has a **capacity**. *This means that we can regard them as directed weighted graphs.*

Directed graphs

Definition

A **digraph** *is a pair* $(\mathcal{V}, \mathcal{E})$ *of sets, where* $\mathcal{V}$ *is nonempty and each element of* $\mathcal{E}$ *is an ordered pair of distinct elements of* $\mathcal{V}$*. The elements of* $\mathcal{V}$ *are still called* **vertices**, *while the elements of* $\mathcal{E}$ *are called* **arcs**.

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Remark

In other words, a digraph is a graph where each edge is assigned a direction, usually labeled by an arrow.

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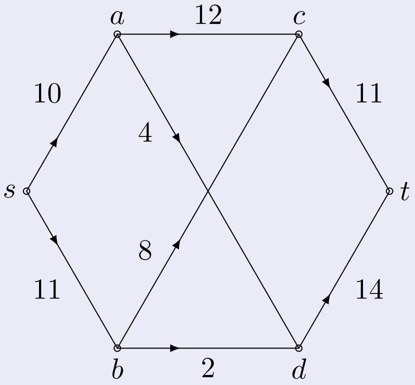
Remark

Sometimes it is also useful to allow non-integer weights, but in this lecture we only consider integer weights.

Example of directed network

Example



A directed network with source s and sink t.

Flows

Definition

Given a directed network with vertex set $\mathcal{V}$, the capacity of arc uv denoted c_{uv}, and given two distinguished vertices s and t, called the **source** *and* **sink**, *respectively, an (s,t)-flow is a set $\mathcal{F}$ of numbers $\{f_{uv}\}$ satisfying*

1. $0 \leq f_{uv} \leq c_{uv}$ *for all* $(u,v) \in \mathcal{E}$;
2. *(conservation of flow)* $\sum_{v \in \mathcal{V}} f_{uv} = \sum_{v \in \mathcal{V}} f_{vu}$ *for all* $u \in \mathcal{V} - \{s,t\}$.

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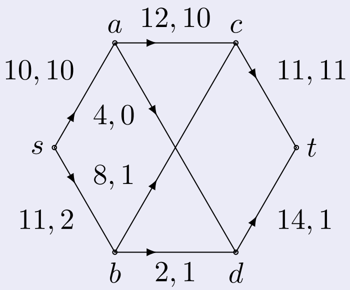
Remark

The conservation of flow means that at any other vertex than s, t of the network, the influx and efflux are equal.

Example of a flow

Example



An (s, t)*-flow. Numbers are* (c_{uv}, f_{uv}).

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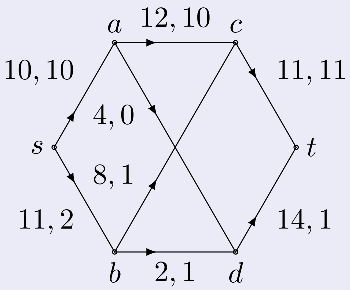

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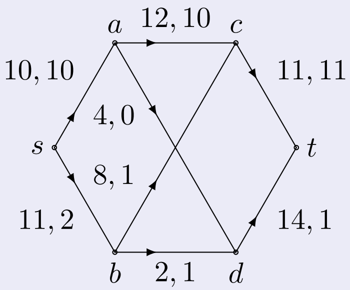

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influx: $f_{ac} + f_{bc} = 10 + 1 = 11$;
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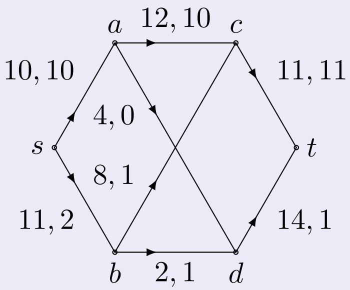

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Let's check the conservation of flow at c:
influx: $f_{ac}+f_{bc} = 10+1 = 11$;
efflux: $f_{ct} = 11$.
The efflux at s:

$$f_{sa} + f_{sb} = 10 + 2 = 12.$$

Efflux at source equals to influx at sink

Proposition

Let s be the source and t be the sink of a flow $\mathcal{F} = \{f_{uv}\}$, then

$$\sum_{v \in \mathcal{V}} f_{sv} - \sum_{v \in \mathcal{V}} f_{vs} = \sum_{v \in \mathcal{V}} f_{vt} - \sum_{v \in \mathcal{V}} f_{tv}.$$

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Proof.

Note that

$$\sum_{u,v \in \mathcal{V}} f_{uv} = \sum_{u,v \in \mathcal{V}} f_{vu}.$$

Then it follows from the conservation of flow properties. □

Value of flow

Remark

The number 12 *in the previous example seems important. We have a name for it.*

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Definition

The **value of a flow** $\mathcal{F} = \{f_{uv}\}$ *in a directed network with source* s*, sink* t *and vertex set* $\mathcal{V}$ *is the integer*

$$val(\mathcal{F}) = \sum_{v \in \mathcal{V}} f_{sv} - \sum_{v \in \mathcal{V}} f_{vs}.$$

Equivalently,

$$val(\mathcal{F}) = \sum_{v \in \mathcal{V}} f_{vt} - \sum_{v \in \mathcal{V}} f_{tv}.$$

Cuts

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Usually we want to maximize the value of a flow. But apparently it is bounded by the capacity of the arcs.

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Definition

*Let $\mathcal{V}$ be the vertex set of a directed network with source s and sink t. An (s,t)-**cut** is a partition $\{\mathcal{S}, \mathcal{T}\}$ of $\mathcal{V}$ such that $s \in \mathcal{S}$ and $t \in \mathcal{T}$. The **capacity** of an (s,t)-cut $\{\mathcal{S}, \mathcal{T}\}$ is the sum of the capacities of all arcs from $\mathcal{S}$ to $\mathcal{T}$, denoted $cap(\mathcal{S}, \mathcal{T})$:*

$$cap(\mathcal{S}, \mathcal{T}) = \sum_{u \in \mathcal{S}, v \in \mathcal{T}} c_{uv}.$$

An inequality

Remark

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Theorem

In any directed network, the value of an (s,t)*-flow never exceeds the capacity of any* (s,t)*-cut.*

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Intuitively, the capacity of a (s, t)-cut implies the maximal possible amount of flow from $\mathcal{S}$ to $\mathcal{T}$.

Theorem

In any directed network, the value of an (s, t)-flow never exceeds the capacity of any (s, t)-cut.

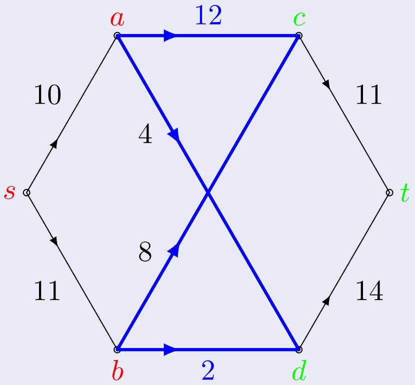
Sketch of proof.

$$val(\mathcal{F}) = \sum_{u \in \mathcal{S}, v \in \mathcal{T}} (f_{uv} - f_{vu}).$$

The former terms are at most c_{uv}, the latter terms are at least 0. □

Example of a cut

Example



An (s,t)*-cut with* $\mathcal{S} = \{s, a, b\}$ *and* $\mathcal{S} = \{t, c, d\}$.

Example of a cut

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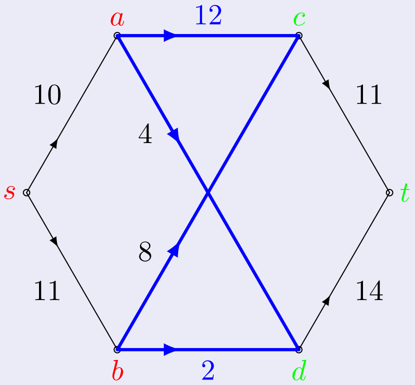

An (s,t)*-cut with* $\mathcal{S} = \{s, a, b\}$ *and* $\mathcal{S} = \{t, c, d\}$.

$cap(\mathcal{S}, \mathcal{T}) = 12 + 2 + 4 + 8 = 26.$

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Now we know that the capacity of any cut is an upper bound of a flow. So the value is at most the minimal capacity of all (s, t)-cuts. But is it always attainable?

Fortunately, the answer is yes!

Theorem (Max flow-Min cut)

In a directed network, the maximal value of an (s, t)-flow equals the minimum capacity of all (s, t)-cuts.

Improving flows via chains

Remark

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We can simply increase the flow via some arcs. Since it is a flow, those arcs must be linked from s to t. Regardless of the directions of arcs, we would call a trail by the term **chain**.

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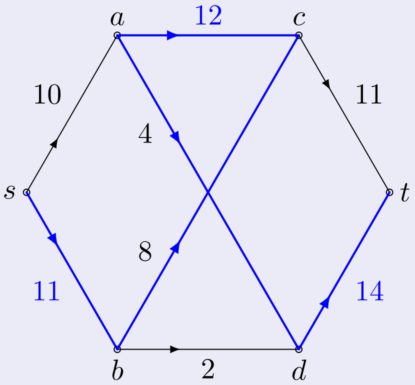
Definition

In a chain, if an arc has the same direction as the chain, it is a **forward arc**; *otherwise it is a* **backward arc**.

Example: a chain

Example

$sbcadt$ is a chain from s to t.



Forward arcs are sb, bc, ad, dt; backward arc is ac.

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Definition

A chain is **flow-augmenting** *if* $f_{uv} < c_{uv}$ *for all forward arcs* uv *and* $f_{uv} > 0$ *for all backward arcs* uv.

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Definition

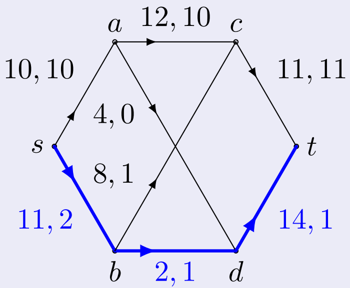
A chain is **flow-augmenting** *if $f_{uv} < c_{uv}$ for all forward arcs uv and $f_{uv} > 0$ for all backward arcs uv.*

Remark

We want more flow in the direction of the chain. So for forward arcs we want higher f_{uv}, while for backward arcs we want to reduce the value of f_{uv}.

Example: improvement via a flow-augmenting chain

Example



An (s,t)*-flow with value* 12*.*
$sbdt$ *is a flow augmenting chain with maximal amount of improvement*
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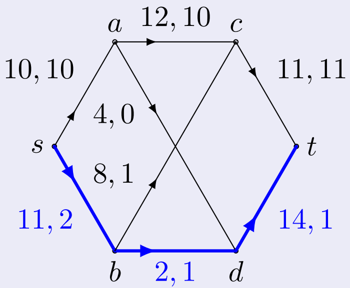
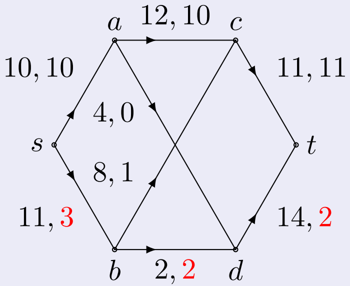


An (s, t)-flow with value 12. $sbdt$ is a flow augmenting chain with maximal amount of improvement $\min(11-2, 2-1, 14-1) = 1$.



The max adjustment is the minimum of all $c_{uv} - f_{uv}$ for forward arcs uv and f_{uv} for all backward arcs uv. We adjust 1 along the chain and get a flow of value 13.

Sketch of proof

Proof of the Theorem.

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If a flow-augmenting chain exists, then the maximal value is not attained yet. So for any maximal flow $\mathcal{F}$, no flow-augmenting chain exists. Now let $\mathcal{S}$ be the set of vertices that there is a flow-augmenting chain from s to it. And let $\mathcal{T}$ be the complement of $\mathcal{S}$. Note that $s \in \mathcal{S}$, and $t \notin \mathcal{S}$, so $\{\mathcal{S}, \mathcal{T}\}$ is an (s, t)-cut.For any $u \in \mathcal{S}$ and $v \in \mathcal{T}$. There is a flow-augment chain from s to u but no such chain from s to v. Then if there are arcs uv or vu, we must have

$$f_{uv} = c_{uv}, f_{vu} = 0.$$

Hence $\mathcal{F}$ reaches the maximal capacity with respect to the cut $\{\mathcal{S}, \mathcal{T}\}$, and as a result $val(\mathcal{F}) = cap(\mathcal{S}, \mathcal{T})$. □

The maximal flow of our example

Remark

The min-cut is $\{s, a, b, c\}, \{t, d\}$ *with capacity* $4 + 2 + 11 = 17$.

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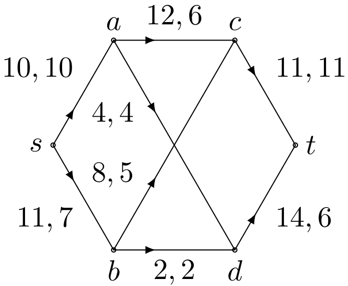

A maximal (s, t)-flow. Numbers are (c_{uv}, f_{uv}).

The End

Thank you very much!
Please do the CIOS in time.