Math 2603 - Lecture 3
Section 1.3, Section 2.1 & 2.2 Sets

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Logical arguments

Rules of inference

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1. Modus ponens;
2. Modus tollens;
3. Disjunctive syllogism;
4. Chain rule;
5. Resolution.

Modus ponens

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Definition

Modus ponens *is the following valid argument:*

$$\begin{array}{c} p \\ p \to q \\ \hline q \end{array}$$

Remark

In Latin, 'modus ponens' means 'method of affirming'. And it is indeed an important way to affirm a statement.

Validity of modus ponens

Even though it is so straight forward, we still need to verify the validity of modus ponens. We can write its truth table

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Proof.

	Premise #1	Premise #2	Conclusion
q	p	$p \to q$	q
T	T	T	T
T	F	T	T
F	T	F	F
F	F	T	F

The only critical row is the first row, and modus ponens is valid. $\square$

Modus tollens

Definition

Modus tollens *is the following valid argument:*

$$\begin{array}{c} p \to q \\ \neg q \\ \hline \neg p \end{array}$$

Modus tollens

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Modus tollens *is the following valid argument:*

$$\begin{array}{c} p \rightarrow q \\ \neg q \\ \hline \neg p \end{array}$$

Remark

Since $p \rightarrow q$ *is logically equivalent to its contrapositive* $\neg q \rightarrow \neg p$*, its validity follows from modus ponens.*

Disjunctive syllogism

Definition

Disjunctive syllogism *is the following valid argument:*

$$\begin{array}{c} p \vee q \\ \neg p \\ \hline q \end{array}$$

Chain rule

Definition

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$$\begin{array}{c} p \to q \\ q \to r \\ \hline p \to r \end{array}$$

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Remark

Chain rule is related to modus ponens.

Resolution

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Remark

Validity: we prove by contradiction. Suppose premises are true and $p \vee q$ *is false, then both* p *and* q *are false. Since* $p \vee r$ *is true,* r *must be true; since* $q \vee \neg r$ *is true,* $\neg r$ *must be true, a contradiction!*

Sets

What is a set

Set is probably one of the most important mathematical notions. Roughly speaking, a set is a collection of objects. And maybe surprisingly, it is so fundamental that we are unable to give a rigorous definition of sets.

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Let S be a set. If an object x belongs to S, we write $x \in S$ and x is called an **element** *of the set S; otherwise we write $x \notin S$.*

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Axiom

The **axiom of extension** *says that a set is completely determined by what its elements are - not the order in which they might be listed or the fact that some elements might be listed more than once.*

Set-roster notation

If we can list all elements $x_1, x_2, \ldots$ of a set S, we can use the following notation:

$$S = \{x_1, x_2, \ldots\}.$$

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Remark

Set-roster notation works no matter whether a set contains finitely many or infinitely many elements, while in the latter case we should make the pattern of elements very clear as it is impossible to present all of them.

Set-builder notation

If a set contains too many elements, or it is defined implicitly by some properties, we may use another notation. Let S be a set and let $P(x)$ be a property (predicate) that elements of S may or may not satisfy. We may define a new set to be the set of all elements x in S such that $P(x)$ is true. We denote this set as follows:

$$\{x \in S \mid P(x)\}.$$

This notation is called the **set-builder notation**.

Remark

In set-builder notation, within the pair of curly brackets, the space is divided into two parts by a vertical bar $|$. *On the left is the description of a general element of the set, and on the right is the properties that all elements must satisfy.*

Notations of some important sets

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Symbol	Meaning
$\mathbb{Z}$	the set of all integers
$\mathbb{N}$	the set of all positive integers
$\mathbb{Q}$	the set of all rational numbers
$\mathbb{R}$	the set of all real numbers
$\mathbb{C}$	the set of all complex numbers

Example of notations of sets

Example

Denote the set of all even integers not exceeding 10 *using both set-roster notation and set-builder notation.*

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Solution

Set-roster notation: $\{2, 4, 6, 8, 10\}$.
Set-builder notation:

$$\{x \in \mathbb{N} \mid x \text{ is even } \& \; x \leq 10\}$$

or

$$\{2k \mid k \in \mathbb{Z}, 1 \leq k \leq 5\}.$$

Equality of sets

Definition

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$$A = B \leftrightarrow (\forall x,\ x \in A \leftrightarrow x \in B)\,.$$

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Remark

Reminder: the order of elements in a set does not matter.

Subsets

Definition

If A and B are sets, then A is called a **subset** *of B, written $A \subseteq B$, if and only if every element of A is also an element of B; otherwise we write $A \not\subseteq B$. If $A \subseteq B$, we also write $B \supseteq A$.*

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Definition

If A and B are sets, then A is called a **proper subset** *of B, written $A \subsetneqq B$ (LaTeX symbol \subsetneqq), if and only if every element of A is in B but there is at least one element of B that is not in A. In order words, $A \subsetneqq B$ if and only if $A \subseteq B$ and $A \neq B$.*

Example of subsets

Example

Among the sets

$$\mathbb{N}, \mathbb{Q}, \mathbb{R}, \mathbb{Z}, \mathbb{C}$$

find as many as possible pairs of A and B such that $A \subseteq B$.

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Solution

We have

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}.$$

Empty set

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Remark

For all set A, $\emptyset \subseteq A$. $\emptyset$ is very special that it could make a lot of statements with $\forall$ false. So when checking the truth value of statements, do not forget this set!

Power sets

Definition

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Remark

$\emptyset$ belongs to the power set of any set. When A is infinite, $\mathcal{P}(A)$ is infinite too.

Operations on sets

Definitions

Definition

Let A and B be two subsets of a universal set U.

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- *The* **difference** *of B minus A, denoted $B \backslash A$, is the set of all elements that are in B and not in A.*
- *The* **complement** *of A, denoted A^c, is the set of all elements in U that are not in A.*
- *The* **symmetric difference** *of A and B, denoted $A \oplus B$, is the set of all elements that belong to exactly one of A, B.*

Formal definition

Remark

In symbols, we have

$$A \cup B = \{x \in U \mid x \in A \text{ or } x \in B\}.$$

$$A \cap B = \{x \in U \mid x \in A, x \in B\}.$$

$$B \backslash A = \{x \in B \mid x \notin A\}.$$

$$A^c = \{x \in U \mid x \notin A\} = U \backslash A.$$

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$$A \oplus B = \{x \in U \mid x \in A \backslash B \text{ or } x \in B \backslash A\}.$$

Remark

The complement is a little special that we need to fix a universal set before talking about it.

Example: operations

Example

Let $U = \{1, 2, 3, 4, 5\}, A = \{2, 4\}$ *and* $B = \{2, 3, 5\}$.

(a) *Find* $A \cup B$.

(b) *Find* $B \backslash A$.

(c) *Find* B^c.

(d) *Find* $(A \cap B)^c$.

(e) *Find* $A \oplus B$.

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Solution

(a) $A \cup B = \{2, 3, 4, 5\}$.

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(e) *Find* $A \oplus B$.

Solution

(a) $A \cup B = \{2,3,4,5\}$. *(b)* $B \backslash A = \{3,5\}$.

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Let $U = \{1, 2, 3, 4, 5\}$, $A = \{2, 4\}$ *and* $B = \{2, 3, 5\}$.

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d. *Find* $(A \cap B)^c$.
e. *Find* $A \oplus B$.

Solution

(a) $A \cup B = \{2, 3, 4, 5\}$. *(b)* $B \backslash A = \{3, 5\}$. *(c)* $B^c = \{1, 4\}$.

Example: operations

Example

Let $U = \{1, 2, 3, 4, 5\}, A = \{2, 4\}$ and $B = \{2, 3, 5\}$.

(a) Find $A \cup B$.

(b) Find $B \setminus A$.

(c) Find B^c.

(d) Find $(A \cap B)^c$.

(e) Find $A \oplus B$.

Solution

(a) $A \cup B = \{2, 3, 4, 5\}$. (b) $B \setminus A = \{3, 5\}$. (c) $B^c = \{1, 4\}$. (d) $A \cap B = \{2\}$, so $(A \cap B)^c = \{1, 3, 4, 5\}$

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Let $U = \{1, 2, 3, 4, 5\}$, $A = \{2, 4\}$ *and* $B = \{2, 3, 5\}$.

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(e) *Find* $A \oplus B$.

Solution

(a) $A \cup B = \{2, 3, 4, 5\}$. *(b)* $B \backslash A = \{3, 5\}$. *(c)* $B^c = \{1, 4\}$. *(d)* $A \cap B = \{2\}$, *so* $(A \cap B)^c = \{1, 3, 4, 5\}$ *(e)* $A \backslash B = \{4\}$, *so* $A \oplus B = \{3, 4, 5\}$.

Venn diagrams

If sets A and B are represented as regions in the plane, relationships between A and B can be represented by pictures, called **Venn diagrams**, that were introduced by the British mathematician John Venn (1834-1923) in 1881.

Diagram of subsets

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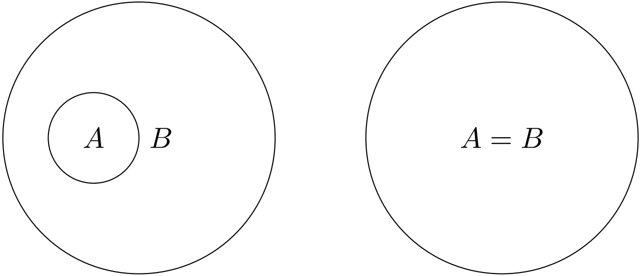

Figure: Venn diagrams for $A \subseteq B$

Diagram of other related sets

The following pictures shows union, intersection and complement of sets.

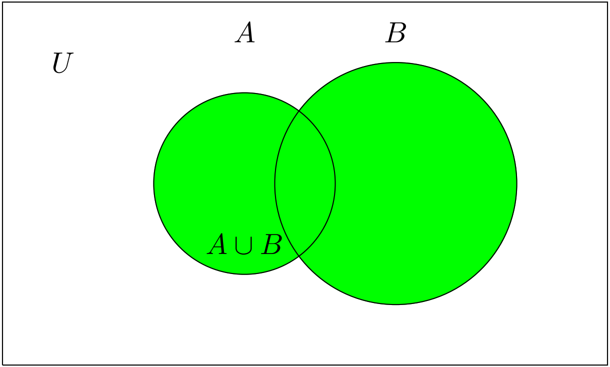

Figure: The set $A \cup B$.

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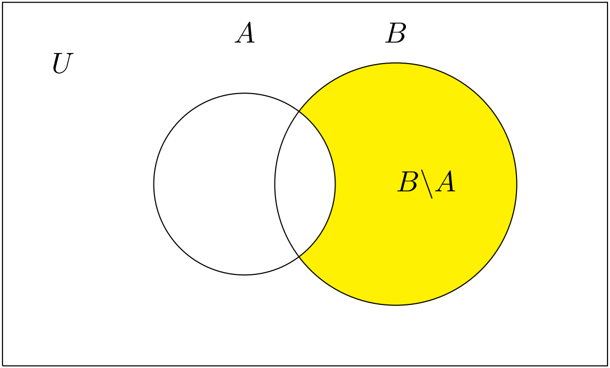

Figure: The set $B \backslash A$.

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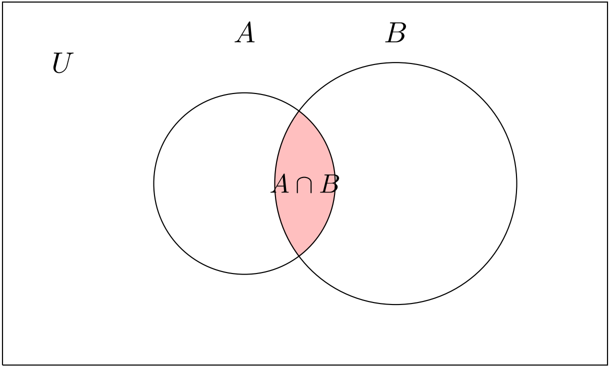

Figure: The set $A \cap B$.

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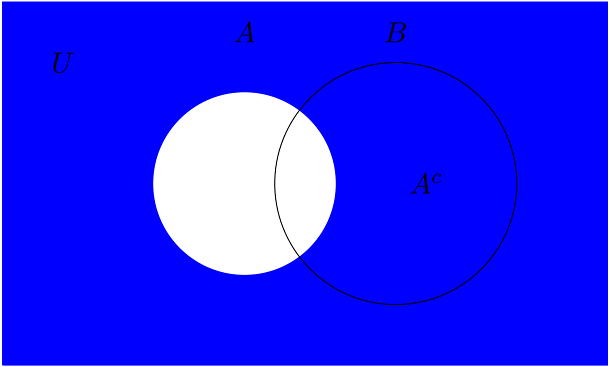

Figure: The set A^c.

HW Assignment #2 - Section 2.1 & 2.2

Section 2.1 Exercise 1(b)(d), 7(b)(c)(g), 12(b)(e)(f).
Section 2.2 Exercise 1(a), 2(a), 3(b)(c), 15(a), 19(a), 30(b)(d).