Math 2603 - Lecture 4
Section 2.3 & 2.4 Relations

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Cartesian Products

Cartesian products

Definition

Given elements a and b, the symbol (a, b) denotes the **ordered pair** *consisting of a and b with the order that a is the first element of the pair and b is the second element. Two ordered pairs (a, b) and (c, d) are equal if and only if $a = c$ and $b = d$.*

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Definition

Given sets A and B, the **Cartesian product** *of A and B, denoted $A \times B$ and read A cross B, is the set of all ordered pairs (a, b), where a is in A and b is in B:*

$$A \times B = \{(a, b) \mid a \in A, b \in B\}.$$

René Descartes

René Descartes (1596-1650) was a French philosopher, mathematician, and scientist. He made a lot of contributions to mathematics. For example, the **Cartesian coordinate system** is named after him.

Example of a Cartesian product

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Let $A = \{2, 4, 6\}$ *and* $B = \{1, 3, 5\}$. *Denote their Cartesian product* $A \times B$ *and find the number of elements in it.*

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Solution

$$A \times B = \{(2,1), (2,3), (2,5), (4,1), (4,3), (4,5), (6,1), (6,3), (6,5)\}.$$

So there are $3 \cdot 3 = 9$ *elements in their Cartesian product.*

Cartesian products of multiple sets

Definition

For $n \geq 2$ *sets* $A_1, A_2, \dots, A_n$*, their Cartesian product is*

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \dots, a_n) \mid a_i \in A_i \text{ for } 1 \leq i \leq n\}.$$

Binary relations

Relations

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Remark

Suppose $\mathcal{R}$ is a relation from A to B, and the pair (a, b) is in $\mathcal{R}$, we write $(a, b) \in \mathcal{R}$, while other people prefer to write $a\mathcal{R}b$.

Example of relations

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Example

If A is the set of GT students registered in Fall 2019 and B be the set of GT courses in Fall 2019. Then

$$\{(a,b) \in A \times B \mid a \text{ is enrolled in } b\}$$

is a relation from A to B.

Properties of relations on A

When we study the properties of a set A, we may need to consider relations on it. In particular, we consider the following properties:

- reflexivity;
- symmetry;
- antisymmetry;
- transitivity.

Reflexive relations

Definition

The relation $\mathcal{R}$ on A is **reflexive**, *if for all $x \in A$, $(x, x) \in \mathcal{R}$.*

Symmetric relations

Definition

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Example

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Solution

Yes. Because the implications are all true.

Antisymmetric relations

Definition

The relation $\mathcal{R}$ *on* A *is* **antisymmetric**, *if for all* $x, y \in A$, $(x, y) \in \mathcal{R}$ *and* $(y, x) \in \mathcal{R}$ *implies* $x = y$.

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Example

The relation $\{(x, y) \in \mathbb{R}^2 \mid x \leq y\}$ is antisymmetric, as $x \leq y$ and $y \leq x$ would imply $x = y$.

Transitive relations

Definition

The relation $\mathcal{R}$ on A is **transitive**, *if for all $x, y, z \in A$, $(x, y) \in \mathcal{R}$ and $(y, z) \in \mathcal{R}$ implies $(x, z) \in \mathcal{R}$.*

Exercise: the properties are independent of each other

Example

Find a relation $\mathcal{R}$ on $\mathbb{N}$ that is reflexive and symmetric, but not transitive.

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Solution

We let $\mathcal{R}$ be a relation on $\mathbb{N}$ such that for all $x, y \in \mathbb{N}$, $(x, y) \in \mathcal{R} \Leftrightarrow |x - y| \leq 1$. This R is reflexive because for any $x \in \mathbb{N}$, $|x - x| = 0 \leq 1$ and if $|x - y| \leq 1$, then $|y - x| \leq 1$.

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We let $\mathcal{R}$ be a relation on $\mathbb{N}$ such that for all $x, y \in \mathbb{N}$, $(x, y) \in \mathcal{R} \Leftrightarrow |x - y| \leq 1$. This R is reflexive because for any $x \in \mathbb{N}$, $|x - x| = 0 \leq 1$ and if $|x - y| \leq 1$, then $|y - x| \leq 1$. However we have $(1, 2) \in \mathcal{R}, (2, 3) \in \mathcal{R}$, but $(1, 3) \notin \mathcal{R}$. So $\mathcal{R}$ is not transitive.

Exercise: examples of relations

Example

Consider the following relations $\mathcal{R}$ defined on $\mathbb{N}$, are they reflexive, symmetric and transitive?

(a) *For all $x, y \in \mathbb{N}$, $(x, y) \in \mathcal{R}$ if and only if $\frac{y}{x} \in \mathbb{N}$.*

(b) *For all $x, y \in \mathbb{N}$, $(x, y) \in \mathcal{R}$ if and only if $x < y$.*

(c) *For all $x, y \in \mathbb{N}$, $(x, y) \in \mathcal{R}$ if and only if $x + y$ is even.*

Exercise: examples of relations

Solution

Relation	*Reflexive*	*Symmetric*	*Transitive*
(a)	*Yes*	*No* $(x = 1, y = 2)$	*Yes*

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Solution

Relation	*Reflexive*	*Symmetric*	*Transitive*
(a)	*Yes*	*No* $(x = 1, y = 2)$	*Yes*
(b)	*No* $(x = 1)$	*No* $(x = 1, y = 2)$	*Yes*

Exercise: examples of relations

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Relation	*Reflexive*	*Symmetric*	*Transitive*
(a)	*Yes*	*No* $(x = 1, y = 2)$	*Yes*
(b)	*No* $(x = 1)$	*No* $(x = 1, y = 2)$	*Yes*
(c)	*Yes*	*Yes*	*Yes*

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Remark

For equivalence relation $\mathcal{R}$ and $(a, b) \in \mathcal{R}$, we usually write $a \sim b$.

Examples of equivalence relations

- On any set of numbers, "being equal" is always an equivalence relation.

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- On any set of numbers, "being equal" is always an equivalence relation.
- Fix $n \in \mathbb{N}$. On any subset of $\mathbb{Z}$, the relation containing all (x, y) when $n \mid (x - y)$ is an equivalence relation.
- On any set of sets, "having the same number of elements (**cardinality**)" is an equivalence relation.

Equivalence classes

Definition

Suppose A is a set and $\mathcal{R}$ is an equivalence relation on A. For each element $a \in A$, the **equivalence class** *of a, denoted $\overline{a}$ and called the class of a for short, is the set of all elements $x \in A$ such that $(x, a) \in \mathcal{R}$. In symbols:*

$$\overline{a} = \{x \in A \mid (x, a) \in \mathcal{R}\}.$$

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Definition

The set of equivalence class of A is called the **quotient set** *of A $\mod \sim$ and denoted $A/\sim$.*

Equivalence classes form a partition

Proposition

Let $\mathcal{R}$ be an equivalence relation on a set A, and $\overline{a}, \overline{b}$ are two equivalence classes. Then, either $\overline{a} = \overline{b}$ or $\overline{a} \cap \overline{b} = \emptyset$.

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Proof.

We prove by cases.
If $(a, b) \in \mathcal{R}$, by symmetry, $(b, a) \in \mathcal{R}$. Then for any $c \in \overline{a}$, since $(c, a) \in \mathcal{R}$ and $\mathcal{R}$ is transitive, we have $(c, b) \in \mathcal{R}$ and $c \in \overline{b}$; similarly for $c \in \overline{b}$, since $(c, b) \in \mathcal{R}$ and $\mathcal{R}$ is transitive, we have $(c, a) \in \mathcal{R}$ and $c \in \overline{a}$. So $\overline{a} = \overline{b}$.

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If $(a, b) \notin \mathcal{R}$. We claim that $\overline{a} \cap \overline{b} = \emptyset$. Suppose $\exists c \in \overline{a} \cap \overline{b}$, then $(c, a), (c, b) \in \mathcal{R}$. By symmetry of $\mathcal{R}$, $(a, c) \in \mathcal{R}$. By transitivity, $(a, b) \in \mathcal{R}$, a contradiction! $\square$

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Definition

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Theorem

The equivalence classes associated with an equivalence relation on A form a partition on A.

Representatives of equivalence classes

Definition

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Remark

By the previous exercise, any element in the same equivalence class serves as its representative.

HW Assignment #2 - Section 2.3 & 2.4

Section 2.3 Exercise 3(b)(c), 5(b)(d), 7, 11(a)(b).
Section 2.4 Exercise 4, 6, 12, 18(a)(e).