Math 2603 - Lecture 5
Section 3.1 & 3.2 Functions

Bo Lin

September 3rd, 2019

Functions

Definition

Definition

*A **function** f from a set A to a set B, denoted $f : A \to B$, is a binary relation from the **domain** A to the **target** B such that every element a in A is related to a unique element in B. If we call this element b, then we say that "f sends a to b" or "f maps a to b", and write $a \xrightarrow{f} b$ or $f : a \to b$. The unique element b to which f sends a is denoted $f(a)$ and called "f of a" or "the value of f at a".*

Definition

Definition

A **function** f *from a set* A *to a set* B*, denoted* $f : A \to B$*, is a binary relation from the* **domain** A *to the* **target** B *such that every element* a *in* A *is related to a unique element in* B*. If we call this element* b*, then we say that "*f *sends* a *to* b*" or "*f *maps* a *to* b*", and write* $a \xrightarrow{f} b$ *or* $f : a \to b$*. The unique element* b *to which* f *sends* a *is denoted* $f(a)$ *and called "*f *of* a*" or "the value of* f *at* a*".*

Remark

In other words, $\forall\, a \in A$*,* $\exists$ *exactly one* $b \in B$ *such that* $(a, b) \in f$*.*

Range and preimage

Definition

For a function $f : A \to B$*, the* **range** *(or* **image***) of* f *is the set*

$$\{b \in B \mid b = f(a) \text{ for some } a \in A\}.$$

Range and preimage

Definition

For a function $f : A \to B$, the **range** *(or* **image***) of f is the set*

$$\{b \in B \mid b = f(a) \text{ for some } a \in A\}.$$

Definition

For a function $f : A \to B$ and any $b \in B$, the **preimage** *of b, denoted $f^{-1}(b)$, is the set*

$$\{a \in A \mid f(a) = b\}.$$

Equality of functions

Definition

Two functions f and g are **equal** *if and only if:*

- *they have the same domain D;*
- *for any $a \in D$, $f(a) = g(a)$.*

Equality of functions

Definition

Two functions f and g are **equal** *if and only if:*

- *they have the same domain D;*
- *for any $a \in D$, $f(a) = g(a)$.*

Remark

By definition, equal functions may have different targets. However, they must have the same range.

Example: equal functions

Example

Are the following pairs of functions f and g equal?

(a) $f : \mathbb{R} \to \mathbb{R}$ *with* $f(x) = x$ *for all* $x \in \mathbb{R}$; $g : \mathbb{R} \to \mathbb{R}$ *with* $g(x) = \sqrt{x^2}$ *for all* $x \in \mathbb{R}$.

(b) $f : \mathbb{R} \to \mathbb{R}$ *with* $f(x) = |x|$ *for all* $x \in \mathbb{R}$; $g : \mathbb{R} \to \mathbb{R}$ *with* $g(x) = \sqrt{x^2}$ *for all* $x \in \mathbb{R}$.

Example: equal functions

Solution

(a) $f(-1) = -1$ *while* $g(-1) = \sqrt{(-1)^2} = \sqrt{1} = 1$, *so* $f \neq g$.

Example: equal functions

Solution

(a) $f(-1) = -1$ *while* $g(-1) = \sqrt{(-1)^2} = \sqrt{1} = 1$, *so* $f \neq g$.
(b) Note that for all $x \in \mathbb{R}$, *we have that* $\sqrt{x^2} = |x|$, *so* $f = g$.

Whether a function is well-defined

When we define a function, we need to make sure that each element in the domain is indeed mapped to a unique element in the target.

Whether a function is well-defined

When we define a function, we need to make sure that each element in the domain is indeed mapped to a unique element in the target.
Consider the following relation F between $\mathbb{Q}$ and $\mathbb{Z}$ such that for all $\frac{m}{n} \in \mathbb{Q}$ with $m, n \in \mathbb{Z}$, we let $\left(\frac{m}{n}, m\right) \in F$. Is F a function?

Whether a function is well-defined

When we define a function, we need to make sure that each element in the domain is indeed mapped to a unique element in the target.

Consider the following relation F between $\mathbb{Q}$ and $\mathbb{Z}$ such that for all $\frac{m}{n} \in \mathbb{Q}$ with $m, n \in \mathbb{Z}$, we let $\left(\frac{m}{n}, m\right) \in F$. Is F a function?

The answer is "no". For the element "one half" in $\mathbb{Q}$, it could be expressed as $\frac{1}{2}$. By definition $\left(\frac{1}{2}, 1\right) \in F$. in addition, one half is also equal to $\frac{2}{4}$, so $\left(\frac{1}{2}, 2\right) \in F$. But any element in the domain of a function cannot be mapped to more than one element in the target, so F is not a function.

Properties of Functions

One-to-one property

Definition

Let F be a function from a set A to a set B. F is **one-to-one** *(or* **injective***) if and only if for all elements a_1 and a_2 in A, if $F(a_1) = F(a_2)$, then $a_1 = a_2$. Symbolically,*

$$F : A \to B \text{ is one-to-one}$$
$$\Leftrightarrow \forall a_1, a_2 \in A,\ F(a_1) = F(a_2) \to a_1 = a_2.$$

One-to-one property

Definition

Let F be a function from a set A to a set B. F is **one-to-one** *(or* **injective***) if and only if for all elements a_1 and a_2 in A, if $F(a_1) = F(a_2)$, then $a_1 = a_2$. Symbolically,*

$$F : A \to B \text{ is one-to-one}$$
$$\Leftrightarrow \forall a_1, a_2 \in A,\ F(a_1) = F(a_2) \to a_1 = a_2.$$

Remark

The one-to-one property is equivalent to "different elements in the domain have different images".

Example: one-to-one property

Example

Find out whether the following functions are one-to-one or not.

(a) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 4x - 1$ *for all* $x \in \mathbb{R}$.

(b) $g : \mathbb{Z} \to \mathbb{Z}$, $g(n) = n^2$ *for all* $n \in \mathbb{Z}$.

Example: one-to-one property

Solution

(a) For any two elements $x_1, x_2 \in \mathbb{R}$*, we have*

$$f(x_1) = 4x_1 - 1, f(x_2) = 4x_2 - 1.$$

Suppose $f(x_1) = f(x_2)$*, then* $4x_1 - 1 = 4x_2 - 1$*, and thus* $4x_1 = 4x_2, x_1 = x_2$*. Hence* f *is one-to-one.*

Example: one-to-one property

Solution

(a) For any two elements $x_1, x_2 \in \mathbb{R}$, we have

$$f(x_1) = 4x_1 - 1, f(x_2) = 4x_2 - 1.$$

Suppose $f(x_1) = f(x_2)$, then $4x_1 - 1 = 4x_2 - 1$, and thus $4x_1 = 4x_2, x_1 = x_2$. Hence f is one-to-one.
(b) Note that $g(-1) = g(1) = 1$, so g is not one-to-one.

Onto property

Definition

Let F be a function from a set A to a set B. F is **onto** *(or* **surjective***) if and only if given any element $b \in B$, there exists at least one element $a \in A$ such that $F(a) = b$. Symbolically:*

$$F : A \to B \text{ is onto} \Leftrightarrow \forall b \in B, \exists\, a \in A \text{ such that } F(a) = b.$$

Onto property

Definition

Let F be a function from a set A to a set B. F is **onto** *(or* **surjective***) if and only if given any element $b \in B$, there exists at least one element $a \in A$ such that $F(a) = b$. Symbolically:*

$$F : A \to B \text{ is onto} \Leftrightarrow \forall b \in B, \exists\, a \in A \text{ such that } F(a) = b.$$

Remark

A function is onto if and only if its range equals to its target.

Example: onto property

Example

Find out whether the following functions are onto or not.

1. $f : \mathbb{Z} \to \mathbb{Z}$ *with* $f(n) = 2n + 1$ *for all* $n \in \mathbb{Z}$.
2. $g : \mathbb{Q}^+ \to \mathbb{Q}^+$ *with* $g(x) = \frac{1}{x}$ *for all* $x \in \mathbb{Q}^+$.

Example: onto property

Example

Find out whether the following functions are onto or not.

(a) $f : \mathbb{Z} \to \mathbb{Z}$ *with* $f(n) = 2n + 1$ *for all* $n \in \mathbb{Z}$.

(b) $g : \mathbb{Q}^+ \to \mathbb{Q}^+$ *with* $g(x) = \frac{1}{x}$ *for all* $x \in \mathbb{Q}^+$.

Solution

(a) Note that when $n \in \mathbb{Z}$, $2n + 1$ *is always odd, so all even integers are not in the range of* f *and* f *is not onto.*

Example: onto property

Example

Find out whether the following functions are onto or not.

(a) $f : \mathbb{Z} \to \mathbb{Z}$ *with* $f(n) = 2n + 1$ *for all* $n \in \mathbb{Z}$.

(b) $g : \mathbb{Q}^+ \to \mathbb{Q}^+$ *with* $g(x) = \frac{1}{x}$ *for all* $x \in \mathbb{Q}^+$.

Solution

(a) Note that when $n \in \mathbb{Z}$, $2n + 1$ *is always odd, so all even integers are not in the range of* f *and* f *is not onto.*
(b) For any positive rational number t, $\frac{1}{t}$ *is still a positive rational number. And note that*

$$g\left(\frac{1}{t}\right) = 1/\frac{1}{t} = t.$$

So t *belongs to the range of* g, *and thus* g *is onto.*

One-to-one correspondence

Definition

A **one-to-one correspondence** *(or* **bijection***) from a set A to a set B is a function $F : A \to B$ that is both one-to-one and onto.*

One-to-one correspondence

Definition

A **one-to-one correspondence** *(or* **bijection***) from a set A to a set B is a function $F : A \to B$ that is both one-to-one and onto.*

Remark

Suppose $F : A \to B$ is a bijection and both A and B are finite sets. Then A and B have the same number of elements.

Identity functions

Definition

For any set A, the **identity function** *on A is denoted ι_A (Greek letter Iota), which is defined by*

$$\iota_A(a) = a \quad \forall a \in A.$$

Identity functions

Definition

For any set A, the **identity function** *on A is denoted ι_A (Greek letter Iota), which is defined by*

$$\iota_A(a) = a \quad \forall a \in A.$$

Remark

Identity functions have equal domain and target, and they map every element in the domain to itself. So they are one of the simplest functions.

Floor and ceiling functions

Definition

For a real number x, the **floor function** *of x, denoted by $\lfloor x \rfloor$, is the largest integer that is less than or equals to x; the* **ceiling function** *of x, denoted by $\lceil x \rceil$, is the smallest integer that is greater than or equals to x.*

Floor and ceiling functions

Definition

For a real number x, the **floor function** *of x, denoted by $\lfloor x \rfloor$, is the largest integer that is less than or equals to x; the* **ceiling function** *of x, denoted by $\lceil x \rceil$, is the smallest integer that is greater than or equals to x.*

Example

For $n \in \mathbb{Z}$, $\lfloor n \rfloor = \lceil n \rceil = n$. $\lfloor -1.5 \rfloor = -2$. $\lceil \frac{1}{3} \rceil = 1$.

Floor and ceiling functions

Definition

For a real number x, the **floor function** *of x, denoted by $\lfloor x \rfloor$, is the largest integer that is less than or equals to x; the* **ceiling function** *of x, denoted by $\lceil x \rceil$, is the smallest integer that is greater than or equals to x.*

Example

For $n \in \mathbb{Z}$, $\lfloor n \rfloor = \lceil n \rceil = n$. $\lfloor -1.5 \rfloor = -2$. $\left\lceil \frac{1}{3} \right\rceil = 1$.

Remark

In other words, $\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1$ and $\lceil x \rceil - 1 < x \leq \lceil x \rceil$.

Inverse and Composition

Motivation of inverse functions

Note that the operation by functions are always *directed* - from the domain to the target. Is it possible to reverse the arrows? More specific, can we also define another function that goes from the target to the domain?

Motivation of inverse functions

Note that the operation by functions are always *directed* - from the domain to the target. Is it possible to reverse the arrows? More specific, can we also define another function that goes from the target to the domain?

Remark

The function F should satisfy some properties:

- *for every element in the target of F, it must have at least one preimage - F must be onto;*

Motivation of inverse functions

Note that the operation by functions are always *directed* - from the domain to the target. Is it possible to reverse the arrows? More specific, can we also define another function that goes from the target to the domain?

Remark

The function F should satisfy some properties:

- *for every element in the target of F, it must have at least one preimage - F must be onto;*
- *for every element in the target of F, its preimage cannot have more than 1 element - F must be one-to-one.*

As a result, we can only define inverse functions for bijective functions.

Definition

Definition

Let $F : A \to B$ be a bijection. The **inverse function** *of F, denoted F^{-1}, is a function from B to A with the following property: for each $b \in B$, since F is a bijection, there is a unique element $a \in A$ such that $F(a) = b$, and we let $F^{-1}(b) = a$.*

Definition

Definition

*Let $F : A \to B$ be a bijection. The **inverse function** of F, denoted F^{-1}, is a function from B to A with the following property: for each $b \in B$, since F is a bijection, there is a unique element $a \in A$ such that $F(a) = b$, and we let $F^{-1}(b) = a$.*

Theorem

If $F : A \to B$ is a bijection, so is F^{-1}.

Example: find inverse function

Example

Find the inverse functions of the following bijections:

(a) $f : \mathbb{R} \to \mathbb{R}$ *with* $f(x) = 4x + 1$ *for all* $x \in \mathbb{R}$.

(b) $g : \mathbb{R}^+ \to \mathbb{R}$ *with* $g(x) = \log_2 x$.

Example: find inverse function

Example

Find the inverse functions of the following bijections:

(a) $f : \mathbb{R} \to \mathbb{R}$ *with* $f(x) = 4x + 1$ *for all* $x \in \mathbb{R}$.

(b) $g : \mathbb{R}^+ \to \mathbb{R}$ *with* $g(x) = \log_2 x$.

Solution

(a) For each $y \in \mathbb{R}$, f^{-1} *maps* y *to the unique real number* x *such that* $f(x) = y$. *The key step is to express* x *in terms of* y.

Example: find inverse function

Example

Find the inverse functions of the following bijections:

(a) $f : \mathbb{R} \to \mathbb{R}$ *with* $f(x) = 4x + 1$ *for all* $x \in \mathbb{R}$.

(b) $g : \mathbb{R}^+ \to \mathbb{R}$ *with* $g(x) = \log_2 x$.

Solution

(a) For each $y \in \mathbb{R}$, f^{-1} *maps* y *to the unique real number* x *such that* $f(x) = y$. *The key step is to express* x *in terms of* y. *Note that* $4x + 1 = y$, *so* $x = \frac{y-1}{4}$. *Then the inverse function of* f *is*

$$f^{-1} : \mathbb{R} \to \mathbb{R}, f^{-1}(y) = \frac{y-1}{4} \quad \forall y \in \mathbb{R}.$$

Example: find inverse function

Solution

(b) For each $y \in \mathbb{R}$, $g^{-1}(y)$ *is the unique positive real number* x *such that* $g(x) = y$. *Thus* $\log_2 x = y, 2^y = x$. *Then the inverse function of* g *is*

$$g^{-1} : \mathbb{R} \to \mathbb{R}^+, g^{-1}(y) = 2^y \quad \forall y \in \mathbb{R}.$$

Example: find inverse function

Solution

(b) For each $y \in \mathbb{R}$, $g^{-1}(y)$ *is the unique positive real number* x *such that* $g(x) = y$. *Thus* $\log_2 x = y, 2^y = x$. *Then the inverse function of* g *is*

$$g^{-1} : \mathbb{R} \to \mathbb{R}^+, g^{-1}(y) = 2^y \quad \forall y \in \mathbb{R}.$$

Remark

For bijections, the domain and the target are not always exactly the same set.

Example: find inverse function

Solution

(b) For each $y \in \mathbb{R}$, $g^{-1}(y)$ *is the unique positive real number* x *such that* $g(x) = y$. *Thus* $\log_2 x = y, 2^y = x$. *Then the inverse function of* g *is*

$$g^{-1} : \mathbb{R} \to \mathbb{R}^+, g^{-1}(y) = 2^y \quad \forall y \in \mathbb{R}.$$

Remark

For bijections, the domain and the target are not always exactly the same set.

Remark

Logarithmic functions can be defined as the inverse functions of exponential functions.

The composition of functions

Definition

Let $f : A \to B$ and $g : B \to C$ be two functions. The **composition** *of f and g is another function $g \circ f : A \to C$ such that for every $a \in A$, $(g \circ f)(a) = g(f(a))$. The function $g \circ f$ is read "g composite f" and $g(f(a))$ is read "g of (f of a)".*

The composition of functions

Definition

Let $f : A \to B$ and $g : B \to C$ be two functions. The **composition** *of f and g is another function $g \circ f : A \to C$ such that for every $a \in A$, $(g \circ f)(a) = g(f(a))$. The function $g \circ f$ is read "g composite f" and $g(f(a))$ is read "g of (f of a)".*

Remark

To make sure that $g \circ f$ is well-defined, it suffices to have that the range of f is a subset of the domain of g.

Associativity of composition

Proposition

Let $f : C \to D$, $g : B \to C$ *and* $h : A \to B$ *be functions. Then*

$$f \circ (g \circ h) = (f \circ g) \circ h.$$

Associativity of composition

Proposition

Let $f : C \to D$, $g : B \to C$ and $h : A \to B$ be functions. Then

$$f \circ (g \circ h) = (f \circ g) \circ h.$$

Proof.

By definition, for any $a \in A$, we have

$$(f \circ (g \circ h))(a) = f((g \circ h)(a)) = f(g(h(a))).$$

And

$$((f \circ g) \circ h)(a) = (f \circ g)(h(a)) = f(g(h(a))).$$

In addition, both functions have domain A, so they are equal. □

Composition of inverse functions

Proposition

Functions $f : A \to B$ and $g : B \to A$ are inverse functions of each other if and only if $g \circ f = \iota_A$ and $f \circ g = \iota_B$.

HW Assignment #3 - Section 3.1 & 3.2

Section 3.1 Exercise 10, 15, 17, 26.
Section 3.2 Exercise 2, 8, 9(b)(d), 21, 26.