Math 2603 - Lecture 6
Section 3.3 Bijections and cardinality

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Cardinality

Motivation

Example

Let $X = \{1, 2, 3, 4, 5\}$ *and* $Y = \{a, b, c, d, e\}$. *Do the two sets have equal number of elements?*

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They do, as they both have 5 *elements.*

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Let $X = \{1, 2, 3, 4, 5\}$ *and* $Y = \{a, b, c, d, e\}$. *Do the two sets have equal number of elements?*

Solution

They do, as they both have 5 *elements.*

Remark

There is an alternative way to check: one can map 1 *to* a, 2 *to* b, *and so on. In fact there exists a bijection between elements in* X *and* Y.

Same cardinality

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But what is the cardinality of a set?

Finite and infinite sets

Definition

A set is **finite** *if it is empty, or there exists a bijection between it and the set*

$$\{1, 2, \cdots, n\}$$

of the first n positive integers. If a set is not finite, then it is **infinite**.

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Definition

We define the **cardinality** *of $\emptyset$ as 0, and the cardinality of $\{1, 2, \cdots, n\}$ as n. We denote the cardinality of a set A by $|A|$.*

"Same cardinality" is an equivalence relation

Theorem

For all sets A, B and C, we have the following properties:

- *(reflexivity) A and A have the same cardinality;*
- *(symmetry) if A and B have the same cardinality, then B and A have the same cardinality;*
- *(transitivity) if A and B have the same cardinality and B and C have the same cardinality, then A and C have the same cardinality.*

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Proof.

Reflexivity: ι_A works. Symmetry: let $f : A \to B$ be a one-to-one correspondence, then f^{-1} is also a one-to-one correspondence from B to A. Transitivity: let $f : A \to B$ and $g : B \to C$ be the one-to-one correspondences, then $g \circ f$ is also one-to-one and onto, so it is a one-to-one correspondences from A and C. $\square$

Example: cardinality of Cartesian product

Proposition

Suppose m, n are positive integers. For sets A, B with $|A| = m, |B| = n$, we have $|A \times B| = mn$.

Cardinality of Infinite Sets

A counterintuitive property of infinite sets

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Corollary

An infinite set and its proper subset can have the same cardinality.

Countable sets

Definition

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Note that we can count all elements of $\mathbb{N}$ one by one, we have the following definition.

Definition

If a set and $\mathbb{N}$ *have the same cardinality, then it is* **countably infinite**. *A set is* **countable** *if it is either finite or countably infinite.*

An important question is: what infinite sets are countable?

Exercise: $\mathbb{Z}$ is countable

Example

Show that $\mathbb{Z}$ is countable by constructing a bijection between $\mathbb{N}$ and $\mathbb{Z}$.

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Example

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Proof.

Suppose we can write all integers in a sequence such that every integers appears exactly once in the sequence, then we are done. Because we can let $f : \mathbb{N} \to \mathbb{Z}$ be a function such that $f(n)$ is the n-th term in the sequence. One example of the sequence is:

$$0, 1, -1, 2, -2, 3, -3, \cdots$$

An explicit bijection $f : \mathbb{N} \to \mathbb{Z}$ is:

$$f(n) = (-1)^n \cdot \left\lfloor \frac{n}{2} \right\rfloor.$$

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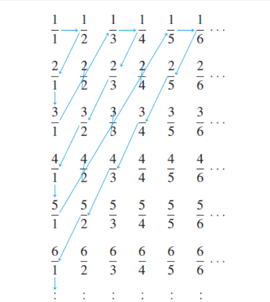

Properties of countable sets

Theorem

The following types of sets are countable:

1. *the subset of a countable set;*
2. *the union of a countable set and a finite set;*
3. *the union of finitely many countable sets;*
4. *the union of countably many countable sets, which means it is the union of an infinite family of sets $S_1, S_2, \cdots$ such that each S_i is a countable set.*

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Corollary

$\mathbb{R}$ *is uncountable.*

Cantor's diagonalization process

Proof.

Suppose we can list all real numbers between 0 and 1 in a sequence $r_1, r_2, r_3, \ldots$. Let the decimal presentation of r_i be

$$0.a_{i1}a_{i2}a_{i3}\ldots$$

Here each a_{ij} is an integer between 0 and 9.

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such that each b_i is different from a_{ii} (there are 10 possible choices of the digit, so this is always doable).

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such that each b_i is different from a_{ii} (there are 10 possible choices of the digit, so this is always doable). Since b is also a real number between 0 and 1, it belongs to the sequence above and thus there exists $n \in \mathbb{N}$ such that $b = r_n$. However, the n-th decimal digit of r_n is a_{nn}, while the n-th decimal digit of b is $b_n \neq a_{nn}$, a contradiction! $\square$

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Proposition

For any real numbers $a < b$*, the open interval*

$$(a, b) = \{x \in \mathbb{R} \mid x > a, x < b\}$$

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Sketch of proof.

The trigonometric function $y = \tan x$ gives a bijection between $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $\mathbb{R}$. And it is easy to find a bijection between any (a, b) and $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. $\square$

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If the cardinality of S is $\aleph$, then the cardinality of $\mathcal{P}(S)$ is usually denoted $2^{\aleph}$. In fact, $2^{\aleph_0} = \aleph_1$.

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There is no largest cardinality of sets.

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Proof.

It suffices to show that there is no bijection between S and $\mathcal{P}(S)$. Suppose a function $\phi : S \to \mathcal{P}(S)$ is a bijection. We consider the following subset of S:

$$T = \{x \in S \mid x \notin \phi(x)\}.$$

Since $T \in \mathcal{P}(S)$, there exists $y \in S$ such that $T = \phi(y)$. Now we check whether $y \in T$.

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Since $T \in \mathcal{P}(S)$, there exists $y \in S$ such that $T = \phi(y)$. Now we check whether $y \in T$. If $y \in T$, by the definition of T, y is an element of S such that $y \notin \phi(y) = T$, a contradiction! Conversely, if $y \notin T$, by the definition of T, y is an element of S such that $y \notin \phi(y)$ does not hold, so $y \in \phi(y) = T$, still a contradiction! $\square$

Relationship between $\aleph_0$ and $\aleph_1$

Theorem

The cardinality of $\mathcal{P}(\mathbb{N})$ *is* $\aleph_1$. *In other words,* $2^{\aleph_0} = \aleph_1$.

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The cardinality of $\mathcal{P}(\mathbb{N})$ is $\aleph_1$. In other words, $2^{\aleph_0} = \aleph_1$.

Proof.

If we use binary representation, all real numbers in the closed interval $[0,1]$ are of the form $0.a_1a_2a_3\ldots$, where each a_i is either 0 or 1. Each sequence of the a_i's corresponds to a subset of $\mathbb{N}$. So $\mathcal{P}(\mathbb{N})$ has the same cardinality as $[0,1]$. It suffices to construct a bijection between $[0,1]$ and $(0,1)$. $\square$

Relationship between $\aleph_0$ and $\aleph_1$

Proof (continued).

Let $f : [0,1] \to (0,1)$ be the following function:

$$f(x) = \begin{cases} \frac{1}{2}, & \text{if } x = 0; \\ \frac{1}{n+2}, & \text{if } x = \frac{1}{n}, n \in \mathbb{N}; \\ x, & \text{otherwise.} \end{cases}$$

Then f is a bijection. $\square$

The continuum hypothesis

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Remark

After hard work by several generations of scholars, it turned out that under our system of axioms, the continuum hypothesis can neither be proved nor disproved, so we can add it or its negation as a new axiom.

HW Assignment #3 - Section 3.3

Section 3.3 Exercise 10, 12(b), 17, 20(c)(d)(f), 21(c).