Math 2603 - Lecture 9
Section 5.1 Mathematical Induction

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The Principle

A motivating example

Example

Show that for every positive integer n, *we have that* $2^n > n$.

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First we look at the case when $n = 1$, $2^1 = 2 > 1$, the statement is true. Then we can check $n = 2, 3, 4, \cdots$. Of course every single case holds, but there are infinitely many cases. What to do? Suppose we already have $2^k > k$ for some indefinite integer k. Then what about $k + 1$? On the left hand side, from k to $k + 1$ the value is doubled, while on the right hand side the value only increases by 1. So we have the following approach: (to be continued) □

A motivating example

Proof continued.

Suppose $2^k > k$. Then

$$2^{k+1} = 2 \cdot 2^k = 2^k + 2^k > k + 2^1 > k + 1.$$

So the statement holds for $k+1$ as well. And we can deduce in a chain: $1 \to 2 \to 3 \to 4 \to \cdots$. This is the pattern of **mathematical induction**. □

The principle

Definition (The principle of mathematical induction)

Let $P(n)$ be a property (predicate) that is defined for integers n, and let a be a fixed integer. Suppose the following two statements are true:

1. *$P(a)$ is true.*
2. *For all integers $k \geq a$, if $P(k)$ is true, then $P(k+1)$ is true.*

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Then the statement "for all integers $n \geq a$, $P(n)$*" is true.*

Remark

In practice, we usually choose $a = 1$ *or* $a = 0$, *while* a *could be any integer.*

Why this principle is true

The principle of mathematical induction is essentially an axiom. Alternatively, it is deduced from the well-ordering property of positive integers, which is also an axiom.

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Let $S = \{k \in \mathbb{Z} \mid k \geq a, P(k) \text{ is false}\}$. We prove by contradiction. Suppose S is nonempty, by the well-ordering principle, S has a least element, say $n \in \mathbb{Z}$. So $P(n)$ is false. Since $P(a)$ is true, $n \neq a$ and thus $n \geq a + 1$.

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An illustrative video of mine

Remark

An illustration: in a sequence of dominoes, if the very first one falls backward, what would happen? They all fall.

At Guinness world records museum (Copenhagen, Denmark)

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Remark

Note that the inductive step is a statement with the universal quantifier. So we usually take a particular but arbitrarily chosen integer $k \geq a$ and suppose that $P(k)$ is true (this is called the **inductive hypothesis***). Finally, we try to draw the conclusion that $P(k+1)$ is true.*

Example: challenges in the inductive step

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The currency of some country only has two denominations of coins - 3 cents and 5 cents. Prove that for integer $n \geq 8$, we can pay exactly n cents using those coins.

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Hint

$$8 = 5 + 3, 9 = 3 + 3 + 3, 10 = 5 + 5, 11 = 5 + 3 + 3, \cdots$$

Proof by cases in the inductive step

Proof.

We use induction on n. The basis step is when $n = 8$. Since $8 = 3 + 5$, it is true. For the inductive step, suppose $k \geq 8$ is an arbitrary integer such that we can obtain k cents using those coins. How to obtain $k + 1$ cents? There are two cases.

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Case 1: at least one 5-cent coin is used. Then we may replace one such coin by two 3-cent coins, so $k + 1$ cents is obtained.

Case 2: no 5-cent coin is used. Since $k \geq 8$, at least three 3-cent coins are used. We can take three of them, and replace them by two 5-cent coins, so $k + 1$ cents is also obtained.

In summary, $k + 1$ cents is also able to obtain and we are done. □

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Remark

This is a typical example for an alternative form of induction - we will discuss it next.

Strong Form and alternative Forms

Motivation: stronger inductive hypothesis

Recall our example of 3- and 5-cents coins. When carrying out the inductive step, it is natural to add a coin, but then the total value of the coins would increase by 3 instead of 1. Then can we modify the principle to deal with this situation?

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Remark

If we think carefully about the pattern, we note that the inductive hypothesis is usually "$P(k)$ is true" for some arbitrary positive integer k. Suppose we begin with $P(1)$. When we already get $P(k)$, we must have got **all** *of $P(1), P(2), P(3), \cdots, P(k-1)$! So it is possible to strengthen the inductive hypothesis.*

The strong form

Definition (Principle of mathematical induction (strong form))

The **strong form** *of mathematical induction is the following:*
Let $P(n)$ be a property that is defined for integers n, and let a and b be fixed integers with $a \leq b$. Suppose the following two statements are true:

1. *(***basis step***) $P(a), P(a+1), \cdots, P(b)$ are all true.*
2. *(***inductive step***) For any integer $k \geq b$, if $P(i)$ is true for all integers $a \leq i \leq k$, then $P(k+1)$ is true.*

Then the statement "for all integers $n \geq a$, $P(n)$" is true. The supposition that $P(i)$ is true for all integers $a \leq i \leq k$ is called the **inductive hypothesis**.

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Example: divisible by a prime

Let's redo the proof of this statement:

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Proof.

Basis step: $n = 2$ is divisible by the prime number 2.
Inductive step: suppose $k \geq 2$ is an integer such that for integers i with $2 \leq i \leq k$, i is divisible by a prime number. Now we consider the case when $n = k + 1$. To analyze the divisors of $k + 1$, we need to know whether it is prime or composite (must be in either case as it is greater than 1). We apply division into cases. (to be continued) □

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Proof continued.

Case 1: $k+1$ is a prime number. Then it is divisible by itself, which is a prime number.

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Case 2: $k+1$ is a composite number. Then there exist positive integers $a, b > 1$ such that $k+1 = ab$. Since $b > 1$, a must be less than $k+1$. So $2 \leq a \leq k$. We apply the inductive hypothesis, then a is divisible by a prime number p. By the transitivity of divisibility, p also divides $k+1$. So our inductive step is done. □

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Remark

If we don't which known case we need as a premise to deduce the next case, we may apply the strong form.

Comparison of the two forms

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The ordinary principle uses "$P(10)$ *implies* $P(11)$*", while the strong form uses "the conjunction of* $P(1), P(2), \cdots, P(10)$ *implies* $P(11)$*". The punchline is that, when we have* $P(10)$*, we must have* **already** *obtained* $P(1), P(2), \cdots, P(9)$ *along the way!*

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In summary, if "$P(10)$ *implies* $P(11)$*" works, we can just apply the ordinary version; otherwise, we may need a stronger inductive hypothesis and apply the strong form.*

Alternative forms - leap by $m > 1$

The following form of mathematical induction is correct too:

Theorem

Let $P(n)$ be a property (predicate) that is defined for integers n, and let a be a fixed integer, $1 < m \in \mathbb{N}$. Suppose the following two statements are true:

1. $P(a), P(a+1), \cdots, P(a+m-1)$ *are true (m cases in total).*
2. *For all integers $k \geq a$, if $P(k)$ is true, then $P(k+m)$ is true.*

Then the statement "for all integers $n \geq a$, $P(n)$" is true.

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Remark

This version is similar to the strong form.

Alternative form - forward-backward

Theorem

Let $P(n)$ be a property (predicate) that is defined for integers n, and let a be a fixed integer. Suppose the following two statements are true:

1. *For $i \in \mathbb{N}$, $P(a_i)$ are true, here $a = a_1 < a_2 < \ldots$ is an infinite sequence of integers.*
2. *For all integers $k \geq a + 1$, if $P(k)$ is true, then $P(k-1)$ is true.*

Then the statement "for all integers $n \geq a$, $P(n)$" is true.

Applications

Closed form of sums

Definition

If a sum with a variable number of terms is shown to be equal to a formula that does not contain either an ellipsis $\cdots$ *or a summation symbol* $\sum_i$, *we say that it is written in* **closed form**.

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Remark

A closed form may be a sum too, but it must contain a constant number of summands.

Example: find closed form of sums

Example

Let $n \in \mathbb{N}$. *Prove that* $\sum_{k=1}^{n}(2k-1) = n^2$.

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Let $n \in \mathbb{N}$*. Prove that* $\sum_{k=1}^{n} (2k-1) = n^2$*.*

Solution

We apply induction on n*. When* $n = 1$*, both sides are* 1*.*

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Let $n \in \mathbb{N}$. *Prove that* $\sum_{k=1}^{n} (2k-1) = n^2$.

Solution

We apply induction on n. *When* $n = 1$, *both sides are* 1. *Suppose the statement holds for* $n = m$, *where* $m \in \mathbb{N}$. *Then* $\sum_{k=1}^{m} (2k-1) = m^2$.

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Solution

We apply induction on n. *When* $n = 1$, *both sides are* 1. *Suppose the statement holds for* $n = m$, *where* $m \in \mathbb{N}$. *Then* $\sum_{k=1}^{m} (2k-1) = m^2$. *Now we consider the case when* $n = m+1$. *It suffices to show that* $\sum_{k=1}^{m+1} (2k-1) = (m+1)^2$. *Finally*

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$$\sum_{k=1}^{m+1}(2k-1) = \sum_{k=1}^{m}(2k-1) + 2(m+1) - 1 = m^2 + 2m + 1 = (m+1)^2.$$

Problem-solving strategy: guess the answer, prove by induction

In many situations, we don't know the answer of a sum. However, if we manage to figure it out, it is usually trivial to prove it by induction.

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Example

Let n be a positive integer. For positive integer k, $k!$ is the factorial of k, which is $\prod_{i=1}^{k} i$. Evaluate the following sum:

$$\sum_{k=1}^{n} (k \cdot k!).$$

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In many situations, we don't know the answer of a sum. However, if we manage to figure it out, it is usually trivial to prove it by induction.

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Let n be a positive integer. For positive integer k, $k!$ is the factorial of k, which is $\prod_{i=1}^{k} i$. Evaluate the following sum:

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Hint

The first few terms in the sequence: $1, 4, 18, 96$.

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Hint

The first few terms in the sequence: $1, 4, 18, 96.$
And the corresponding sums for small n*:* $1, 5, 23, 119.$ *Any pattern?*

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Solution

We claim that the sum is $(n+1)! - 1$*, and we use induction to prove it. When* $n = 1$*, the sum is* 1 *and* $(1+1)! - 1 = 2 - 1 = 1$*. As for the inductive step, suppose* m *is an arbitrary positive integer such that* $\sum_{k=1}^{m} k \cdot k! = (m+1)! - 1$*. Then*

$$\sum_{k=1}^{m+1} k \cdot k! = \sum_{k=1}^{m} k \cdot k! + (m+1) \cdot (m+1)!$$
$$= [(m+1)! - 1] + (m+1) \cdot (m+1)!$$
$$= (m+2) \cdot (m+1)! - 1 = (m+2)! - 1.$$

So the claim is still true when $n = m + 1$*. We are done.*

Example: a flawed proof using induction

Example

Here is a "proof" of the false statement "for all integers $n \geq 1$, $3^n - 2$ *is even."*

Proof.

Suppose the statement is true for an arbitrary integer $k \geq 1$. *Then* $3^k - 2$ *is even. We must show that* $3^{k+1} - 2$ *is even. But*

$$3^{k+1} - 2 = 3^k \cdot 3 - 2 = (3^k - 2) + 2 \cdot 3^k.$$

Now $3^k - 2$ *is even by inductive hypothesis and* $2 \cdot 3^k$ *is even by definition. Hence their sum is also even. It follows that* $3^{k+1} - 2$ *is even, which is what we needed to show.* □

What is the flaw?

Induction makes no sense without basis step

Solution

All steps in the "proof" are correct, but it misses the basis step and in fact the basis step is apparently false.

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Remark

Although the inductive step is usually the essential step in a proof by induction, please note that it is only an implication! So if the premise is false, it is an invalid argument and it is useless. As a result, it is vital to make sure that the basis step is done correctly.

Example: a hidden flaw

Example

Here is a "proof" of the false statement "for all nonzero real numbers r and nonnegative integer n, $r^n = 1$."

Proof.

Fix r, we use strong induction on n. Basis step: when $n = 0$, since $r \neq 0$, $r^0 = 1$ is true.
Inductive step: suppose $k \geq 0$ is an arbitrary integer such that $r^i = 1$ for all $0 \leq i \leq k$. Note that $r^{k+1} = r^{k+k-(k-1)} = r^k \cdot r^k / r^{k-1}$. By the inductive hypothesis, $r^k = r^{k-1} = 1$, so r^{k+1} is also 1. The inductive step is done. □

What is the flaw?

Mind the range of numbers that the hypothesis applies to

Solution

The basis step is absolutely correct. In the inductive step, the formulas are correct. The flaw is that when $k = 0$, $k - 1 = -1$, which is no longer between 0 and k! So in this particular case we don't have $r^{k-1} = 1$!

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The basis step is absolutely correct. In the inductive step, the formulas are correct. The flaw is that when $k = 0$, $k - 1 = -1$, which is no longer between 0 and k! So in this particular case we don't have $r^{k-1} = 1$!

Remark

In this flawed proof, we can see that if we already have that the claim is true for $k = 0, 1$, then the proof works. But this is expected - when $k = 1$, the claim becomes $r = 1$, and if $r = 1$, the statement would be true. Otherwise, it's false.

HW Assignment #5 - today's sections

Section 5.1 Exercise 3, 6(c)(e), 8(a), 11, 13, 40(a).